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SATELLITE MOTION IN AN AXI-SYMMETRIC GRAVITATIONAL FIELD PART I: PERTURBATIONS DUE TO J_2 (SECOND ORDER) AND J_3

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SATELLITE MOTION IN AN AXI-SYMMETRIC GRAVITATIONAL FIELD

PART 1: PERTURBATIONS DUE TO J_2 (SECOND ORDER) AND J_3

by

R. H. Gooding

SUMMARY

Full details are given of the untruncated second-order orbital theory, of which a résumé (including the formulae finally derived) was published in 1983. The analysis only takes account of the first two zonal harmonics, J_2 and J_3 (geopotential assumed), but the extension to an arbitrary zonal harmonic will be covered in a later Report (Part 2).

The principal merit of the theory derives from the extreme compactness of the results obtained for short-period perturbations in position and velocity, these perturbations being expressed relative to a system of spherical-polar coordinates based on a rotating 'mean orbital plane' The use of osculating elements is thereby avoided, and mean elements are defined so as to give the coordinate perturbations their simplest possible form. To make the theory both complete and compact, an intermediate set of elements, described as 'semi-mean', is also required.

A numerical assessment of the theory will be included in another Report (Part 3).

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1 INTRODUCTION

The 'main problem' in modelling the motion of an artificial Earth satellite has always been recognized as the representation of the perturbations due to the low-degree zonal harmonics of the geopotential. A considerable literature has resulted, much of which was surveyed by the present author in earlier papers[1-3]. The last two of these papers gave a second-order theory* for orbits of low eccentricity: Ref 2 was confined to the perturbations due to J_2, but Ref 3 covered the effects (for low e) of the general (tesseral) harmonic, $J_{\ell m}$; as in Ref 1, the formulae associated with $J_{\ell m}$ were shown to be applicable to luni-solar perturbations when ℓ and m are taken as negative and zero respectively. The novelty in Refs 2 and 3 lay in the compactness that could be achieved by expressing the short-period perturbations in terms of a particular system of cylindrical-polar coordinates; however, when the theory was extended to an arbitrary elliptic orbit (with formulae therefore untruncated in e), the corresponding system of spherical-polar coordinates was found to be more appropriate.

The outcome of the extension just referred to has been a complete second-order theory for J_2 and J_3, and a summarized account of the new theory was presented to the 1982 IAF Congress[4]. The present Report gives the full details of the purely analytical part of the theory, including (in particular) the lengthy expressions for the perturbations in the osculating elements as well as the strikingly compact expressions for perturbations in the coordinates. The original intention was to include details of the hybrid (semi-analytical) component[5] of the orbital model, the component that models the long-term evolution of the chosen set of mean orbital elements. However, the resulting paper would have been unduly long, so most of this material has been held over for a later Report. Further, formulae have recently been derived that generalize the J_3-results (and the first-order J_2-results) of the present paper to an arbitrary zonal harmonic (J_ℓ); these formulae are unexpectedly compact and merit priority in publication.

Thus this Report constitutes Part 1 of an intended trilogy. Part 2 (Technical Report 89022) will cover the purely analytical theory of the generalization to J_ℓ; since each J_ℓ, for $\ell > 2$, only needs be covered to first order, the results for different ℓ may be superposed linearly to provide, in combination with the J_2^2-results from Part 1, a complete second-order theory for

* Since all other geopotential coefficients are of order J_2^2, a 'second-order theory' is second-order in J_2, but only has to be first-order in any other J_ℓ covered.

the zonal harmonics. Finally, Part 3 will cover the evolution of mean elements, the topic dropped from Part 1; it is a consequence of the revision in intention for the present paper that a large number of references to 'Part 3' will be found.

Section 2 of the Report provides sufficient background to make the paper essentially independent of the author's previous contributions[1-6] to the subject. Some of the more important notation is introduced in section 2.1, but in view of the complexity of the notation used throughout the paper, a List of Symbols is included at the end of the Report.

Section 3 is devoted to a number of important issues, and in particular to introducing the notion of 'semi-mean elements'. Unlike mean elements, these are not free of short-period variation; their utility arises from the fact that the independent variable of Lagrange's planetary equations has to be transformed from time to true anomaly, to permit an untruncated integration, so each semi-mean element differs from the corresponding mean element by a quantity proportional to the difference between true and mean anomaly. Considerable attention is devoted to the way in which position and velocity are derived (via semi-mean elements) from mean elements, since the whole ethos of the paper is to stress the importance of an algorithm in which (short-period) perturbations are applied directly to a conceptual 'mean' position and velocity. It is essential, of course, that no accuracy should be lost when the algorithm is inverted, so that mean elements can be obtained from position and velocity. This requirement is of particular importance at epoch, and is easier to implement than is sometimes supposed; a method of inverting the algorithm, iteratively, was described in Ref 5. Two aspects of orbital-element singularity are discussed to complete section 3: first, a difficulty that arises in velocity computation but not in position computation; and secondly, difficulties in the long-term propagation of mean elements. The latter discussion is in the nature of an introduction to Part 3.

Specific expressions for the perturbations due to J_2 and J_3, constituting the meat of the Report, are given in sections 4 to 7. Section 4 covers the first-order perturbations due to J_2, in both (osculating) elements and coordinates. Section 5 derives the second-order perturbations, due to J_2 alone, in the elements, and section 6 derives the resulting perturbations in coordinates. Finally, section 7 covers (for both elements and coordinates) the first-order perturbations due to J_3 alone; as already remarked, these are regarded as contributing to the second-order solution of the problem with both J_2 and J_3 present.

At the end of the concluding section, section 8, the most important formulae derived in the Report are referenced for convenience.

2 BACKGROUND

2.1 Osculating elements and related quantities

Our starting point is a standard set of osculating elements for an elliptical orbit, *viz* a (semi-major axis), e (eccentricity), i (inclination), Ω (right ascension of the ascending node), ω (argument of perigee) and M (mean anomaly), all being functions of t (time). The mean motion, n, is directly related to semi-major axis by Kepler's third law,

$$n^2 a^3 = \mu , \tag{1}$$

where μ is the (Earth's) gravitational constant.

We also require, as an intermediary for perturbations in the 'fast' element, M, the quantity σ, known as the modified mean anomaly at epoch (where $t = 0$); this is defined such that (with τ a dummy variable for t)

$$M = \sigma + \int_0^t n \, d\tau , \tag{2}$$

and we rewrite (2) simply as

$$M = \sigma + \int , \tag{3}$$

since this shorthand use of $\int$ will prove very convenient. Despite its name and definition, σ is a *current* variable, determined from M, in principle, by taking the integral of n backwards from t to epoch. It is clearly less accessible than the 'unmodified mean anomaly at epoch', which is determined (also as a current variable) just by $M - nt$, but σ has the advantage that we have, direct from (2),

$$\dot{M} = \dot{\sigma} + n . \tag{4}$$

An important quantity is the true anomaly, v, which is required both in its own right and (in a particular 'mean' form) as an alternative independent variable to t. The derivation of v from M is given in section 3.3, and two

other important quantities may be derived at once when v is known: first, r, the satellite's geocentric distance, is given by

$$p/r = 1 + e \cos v \ , \tag{5}$$

where p is the semi-latus rectum or 'parameter' of the ellipse, defined by

$$p = a(1 - e^2) \ ; \tag{6}$$

second, u, the argument of latitude, is given (and defined) by

$$u = \omega + v \ . \tag{7}$$

We shall so often require to use the difference between the mean and true anomalies as a distinctive quantity that it is useful to have a shorthand for it; the quantity is traditionally known as 'the equation of the centre' and we denote it here by m, so that

$$m = v - M \ . \tag{8}$$

The elements Ω, ω and M, together with some of the other quantities, suffer certain well-known indeterminacies, or 'singularities', for orbits that are close to being circular, equatorial or both. Dealing properly with these singularities has been a major consideration of the study (see section 3.5 and Part 3), but for the moment we simply introduce some further notation. As in Refs 2 and 3, we use ξ and η for the quantities defined by

$$\xi = e \cos \omega \tag{9}$$

and

$$\eta = e \sin \omega \ , \tag{10}$$

these being well-defined for a near-circular orbit that is not also near-equatorial. However, ξ and η are not as significant now as in the earlier papers, which were largely written on the hypothesis of low eccentricity: it was as a corollary of the low-e assumption that U, defined as $M + \omega$, was also appropriately used as a fundamental element, with ξ, η and U replacing e, ω and M, but the use of U is not nearly so effective when the analysis is not truncated in respect of e. In considering near-equatorial orbits, it will

sometimes be convenient, in the same way, to use the triple (ξ, η, ζ) to refer to a set of quantities $(\sin i \sin \Omega, -\sin i \cos \Omega, \cos i)$ that are always well defined (and in fact constitute the direction cosines of the orbital momentum vector in the assumed axis system). Some other quantities – ψ, ρ and L – are conceptually useful in the context of the circular and equatorial singularities, but they are only fully defined at the differential level and their formal introduction is postponed until section 2.2.

As an aid to the concise presentation of formulae, it is convenient now to define seven quantities, as follows:

$$c = \cos i , \tag{11}$$

$$s = \sin i , \tag{12}$$

$$f = s^2 , \tag{13}$$

$$g = 1 - \frac{5}{4} f , \tag{14}$$

$$h = 1 - \frac{3}{2} f , \tag{15}$$

$$q = \left(1 - e^2\right)^{\frac{1}{2}} , \tag{16}$$

(so that $p = aq^2$) and

$$W = q^{-3} (p/r)^2 . \tag{17}$$

It is worth remarking that e and q have the same relation to the so-called 'angle of eccentricity', ϕ, as s and c have to the inclination, i, so that ϕ (defined as $\sin^{-1} e$) might have been taken as a basic orbital element instead of e. Though there is indeed an important parallel between, in particular, e and s (of which there will be a good example in Part 3), there is an essential distinction. Thus, e can exceed unity (hyperbolic orbits) and so remains a satisfactory parameter when ϕ is no longer real (universal parameters, valid for all types of orbit, are considered in Ref 7). It is quite different for s, since as i increases from $\frac{1}{2}\pi$ to π (retrograde orbits) s returns to zero, so that it is an inherently ambiguous quantity; further, s fails to distinguish satisfactorily between distinct near-polar orbits, since its derivative with respect to i is c, which is then close to zero.

The goal of concise presentation is further facilitated by the introduction of the eight families of quantities defined, in principle, by:

$$c_j = \cos(jv + \omega) \,, \qquad s_j = \sin(jv + \omega) \,, \tag{18}$$

$$C_j = \cos(jv + 2\omega) \,, \qquad S_j = \sin(jv + 2\omega) \,, \tag{19}$$

$$\gamma_j = \cos(jv + 3\omega) \,, \qquad \sigma_j = \sin(jv + 3\omega) \,, \tag{20}$$

$$\Gamma_j = \cos(jv + 4\omega) \,, \qquad \Sigma_j = \sin(jv + 4\omega) \,, \tag{21}$$

where j is any (positive, zero or negative) integer. It is to be noted, however, that the arguments of the c_j *etc* will normally not be the osculating v and ω themselves, but the corresponding 'semi-mean' $\tilde{v}$ and $\tilde{\omega}$, to be introduced in section 3.2. Thus the notation $\tilde{c}_j$ *etc* would normally be appropriate, but for simplicity the tildes will be omitted unless (as in parts of section 5) the distinction becomes important.

Finally, it is convenient to introduce here a pair of notation conventions that will be widely used in the long algebraic expressions that are required *en route* to the compact expressions of the final solution. The conventions involve the use of round and square brackets, respectively, to abbreviate polynomials in f and e^2. Thus (j_1, j_2, j_3) stands (in the appropriate context) for $j_1 + j_2 f + j_3 f^2$; $[j_1, j_2]$ stands for $j_1 + j_2 e^2$; and, since the conventions will be combined, $[(j_1, j_2), j_3(j_4, j_5)]$ stands for $j_1 + j_2 f + j_3 e^2(j_4 + j_5 f)$. The j's will always be integers such that, in the last example, j_1 and j_2 are co-prime, and likewise j_4 and j_5.

2.2 Lagrange's planetary equations

Rates of change of osculating elements may be expressed exactly by Lagrange's planetary equations developed in Refs 8 and 9, for example, or (with more detail) in other text-books[10-12]; these assume the existence of a disturbing function, U, that expresses the perturbation potential. The arguments of U are taken to be a, e, i, Ω, ω and M (with t also in principle, but not in the present study); then the six planetary equations are

$$\dot{a} = \frac{2}{na}\frac{\partial U}{\partial M} \,, \tag{22}$$

$$\dot{e} = \frac{1}{na^2e}\left(q^2\frac{\partial U}{\partial M} - q\frac{\partial U}{\partial \omega}\right) \,, \tag{23}$$

$$\dot{i} = \frac{1}{na^2qs}\left(c\frac{\partial U}{\partial \omega} - \frac{\partial U}{\partial \Omega}\right) \,, \tag{24}$$

$$\dot{\Omega} = \frac{1}{na^2qs}\frac{\partial U}{\partial i} , \tag{25}$$

$$\dot{\omega} = \frac{1}{na^2}\left(\frac{q}{e}\frac{\partial U}{\partial e} - \frac{c}{qs}\frac{\partial U}{\partial i}\right) \tag{26}$$

and

$$\dot{\sigma} = -\frac{1}{na^2}\left(\frac{q^2}{e}\frac{\partial U}{\partial e} + 2a\frac{\partial U}{\partial a}\right) . \tag{27}$$

From (27) we at once have $\dot{M}$, in view of (4).

Before we proceed further, it is worth making explicit the conventional assumption, which holds through the paper, that the dot notation refers to total differentiation with respect to time, d/dt, rather than to partial differentiation, $\partial/\partial t$. As the $\partial/\partial t$ notation refers to differentiation whilst all the elements are held constant, *ie* in relation to the osculating orbit, $\partial a/\partial t$ *etc* are trivially zero, and a misunderstanding could hardly arise. For other quantities, however, the distinction can be important: it is inherent in the *concept* of osculating elements that $\dot{r} = \partial r/\partial t$, since $d/dt = \partial/\partial t$ when applied to (position) coordinates relative to fixed axes, but in general the two derivatives are not equal; thus, $\dot{v} \neq \partial v/\partial t$, for example, as appears later in this section.

It will be observed, in (22)-(27), that it is only $\dot{a}$ and $\dot{\Omega}$ that have been expressed in terms of individual partial derivatives, but the other equations can be combined in such a way as to isolate the remaining derivatives, and these combinations can be very useful. Thus from (6) we have

$$\dot{p} = q^2\dot{a} - 2ae\dot{e} , \tag{28}$$

after which (22) and (23) yield

$$\dot{p} = \frac{2q}{na}\frac{\partial U}{\partial \omega} . \tag{29}$$

Again, combination of (24) and (29) gives

$$\frac{d(pc^2)}{dt} = \frac{2qc}{na}\frac{\partial U}{\partial \Omega} , \tag{30}$$

and this is an important result, since, for a disturbing potential that is symmetric with respect to the Earth's polar axis, U is independent of Ω and

hence pc^2 is constant (conservation of $\sqrt{\mu p}\, c$, the polar component of the angular momentum).

Isolation of the two remaining partial derivatives is possible if we introduce, as in previous work[1,3], quantities $\dot{\psi}$ and $\dot{\rho}$ defined by

$$\dot{\psi} = \dot{\omega} + c\dot{\Omega} \tag{31}$$

and

$$\dot{\rho} = \dot{\sigma} + q\dot{\psi} . \tag{32}$$

We then have

$$\dot{\psi} = \frac{q}{na^2e}\frac{\partial U}{\partial e} \tag{33}$$

and

$$\dot{\rho} = -\frac{2}{na}\frac{\partial U}{\partial a} \quad . \tag{34}$$

Though $\dot{\psi}$ and $\dot{\rho}$ are not the rates of change of quantities, ψ and ρ , of the familiar type, since c and q in (31) and (32) are not constant, first-order analysis can be greatly facilitated[1,3] by derivation of perturbations $\delta\psi$ and $\delta\rho$ from integration of (33) and (34), the assumption being that the variation of c and q can be neglected over the period involved. (The informal use of the δ notation here is without prejudice to the formal meaning to be attached from section 3.1 onward.) Clearly, $\delta\psi$ then remains well defined as i approaches zero or π , so long as e does not at the same time approach zero, and this is not in general true for $\delta\omega$ and $\delta\Omega$ individually; again, $\delta\rho$ is invariably well defined, whereas $\delta\sigma$ is in general not so as e approaches zero.

We can regard $\dot{\rho}$ as the non-singular quantity corresponding to $\dot{\sigma}$, and Ref 3 also introduces $\dot{L}$, the non-singular quantity corresponding to $\dot{M}$; thus

$$\dot{L} = \dot{M} + q\dot{\psi} , \tag{35}$$

and again $\dot{L}$ and δL are useful, though there is no familiar integrated 'fast element' L . It would be wrong to give the impression that the integrated quantities ψ, ρ and L are so artificial as to be entirely devoid of meaning, however; it is not hard to see, in particular, that $\dot{\psi}$ is the intrinsic rate of rotation of the perigee direction within the evolving orbital plane, so that ψ

exists as the accumulated angle (with an arbitrary origin) swept out by this direction. It is to be noted that, from (35), using (4) and (32), we also have

$$\dot{L} = \dot{\rho} + n \ . \tag{36}$$

In regard to the potential associated with zonal harmonics only, the right-hand sides of the planetary equations can be expressed as closed functions of true anomaly but not mean anomaly. It is therefore of fundamental importance to be able to transform the independent variable from t, which is effectively equivalent to M, to v, where v is a function just of e and M. We have, in fact,

$$\dot{v} = v_M \dot{M} + v_e \dot{e} \ , \tag{37}$$

where the partial derivatives, denoted by v_M and v_e, are given in (42) at the end of this section; also $\dot{M}$ is given by (4). We can now appreciate the difference between the total time derivative, $\dot{v}$, and the partial derivative, $\partial v/\partial t$, referred to earlier; for the latter we obtain, on replacing $\dot{M}$ and $\dot{e}$, in (37), by n and zero respectively, and setting v_M from (42),

$$\frac{\partial v}{\partial t} = nW \ . \tag{38}$$

Transformation by (38) permits a complete first-order analysis for any zonal harmonic; on proceeding to second order, however, we must allow for the $\dot{e}$ term in (37), and also for the $\dot{\sigma}$ term that is implied by the appearance of $\dot{M}$, both these terms being of first order. More awkwardly, the first-order perturbation in (osculating) v, which is our assumed independent variable, contains short-period terms that are therefore functionally dependent on v itself, and this implies something of a vicious circle; further, some of these terms - see (182) - contain the e^{-1} singularity factor. In second-order analysis for J_2, all the difficulties are overcome if, anticipating the introduction of mean elements (and semi-mean elements) in section 3, we transform (from t) to $\bar{v}$ (or rather $\tilde{v}$) instead of v. We require a new relation to replace (37) and (38), and this turns out to be

$$\dot{\bar{v}} = \bar{n}\bar{W} + O(J_2^2, J_3) \ , \tag{39}$$

since $\bar{n}$ can be defined to include $\dot{\bar{\sigma}}$, whilst $\dot{\bar{e}}$ is $O(J_2^2)$. (The second-order term in (39) can be ignored, for change-of-variable purposes, since the equation is operating on the planetary equations, which are already of at least first order; when $\dot{\bar{v}}$ is required in its own right, however, $\dot{\bar{e}}$ must be allowed for - *cf* (99)).

Application of (38), or (39), to (22)-(27) gives (for any zonal harmonic) first-order perturbations in a, e, i, Ω, ω and σ, as functions of $\bar{v}$. For a complete first-order solution, we still need the perturbation in M, however, and this is complicated by the need, through (3), for the perturbation in $\int$, to add to the perturbation in σ. Taking 'perturbation' and 'integral' to commute, we thus require $\int_0^t \delta n\, d\tau$, where δn is available at once from δa (*cf* (159)) and hence is a function of $\bar{v}$. On applying (39) again, it is immediate that

$$\frac{d(\delta\int)}{d\bar{v}} = \frac{\delta n}{\bar{n}\bar{W}}, \tag{40}$$

where the right-hand side is effectively just a function of $\bar{v}$, since the barred version of (5) can be used to eliminate the implicit presence of $\bar{F}$ in $\bar{W}$. Thus the derivation of $\int_0^t \delta n\, d\tau$ involves just the $\bar{v}$-integration of $\delta n/\bar{n}\bar{W}$.

Evaluation of all the partial derivatives, for substituting in the planetary equations, is straightforward if appeal is made to the following particular derivatives as necessary:

$$r_a = \frac{r}{a}, \qquad r_e = -a\cos v, \qquad r_M = \frac{ae\sin v}{q}; \tag{41}$$

$$u_e = v_e = \frac{\sin v\,(2 + e\cos v)}{q^2}, \qquad u_\omega = 1, \qquad u_M = v_M = W. \tag{42}$$

2.3 Assumed potential

As we are concerned only with the J_2 and J_3 terms of the geopotential, we take as the disturbing function, U, the combination of U_2 and U_3 defined by

$$U_2 = -\frac{\mu}{r} J_2 \left(\frac{R}{r}\right)^2 P_2(\sin\beta) \tag{43}$$

and

$$U_3 = -\frac{\mu}{r} J_3 \left(\frac{R}{r}\right)^3 P_3(\sin\beta) \quad , \tag{44}$$

where β is geocentric latitude, R is the Earth's equatorial radius, and P_2 and P_3 refer to the usual Legendre polynomials.

The Legendre argument, $\sin\beta$, can be eliminated from (43) and (44) by use of the relation (which underlies the name 'argument of latitude' for u)

$$\sin\beta = s \sin u \ , \tag{45}$$

and r can be eliminated by appeal to (5). The analysis can be significantly condensed if we replace J_2 and J_3 by quantities conveniently denoted by K and H, respectively, and defined by

$$K = \frac{3}{2} J_2 \left(\frac{R}{p}\right)^2 \tag{46}$$

and

$$H = \frac{3}{2} J_3 \left(\frac{R}{p}\right)^3 ; \tag{47}$$

the resultant expressions for U_2 and U_3 are then

$$U_2 = \frac{\mu K}{6p} \left(\frac{p}{r}\right)^3 (3fC_2 + 2h) \tag{48}$$

and

$$U_3 = \frac{\mu H}{12p} \left(\frac{p}{r}\right)^4 s\,(5f\sigma_3 + 12gs_1) \ , \tag{49}$$

the notation introduced in (18)-(20) being freely employed.

For many purposes, K and H can be thought of as constants. It is sometimes vital to recognize that they are functionally dependent on p, however, so that, in particular, they have non-zero partial derivatives with respect to a and e; thus the notation $\bar{K}$ and $\bar{H}$ is appropriate when p is replaced by its mean value, $\bar{p}$.

It has already been remarked that for an axi-symmetric disturbing potential, ie for all the Earth's zonal harmonics, not just J_2 and J_3, the polar component of the satellite's angular momentum furnishes a useful constant of the

motion, but a more important constant is provided by the energy integral. The kinetic energy per unit mass of the satellite is given by

$$\text{K.E.} = \mu\left(\frac{1}{r} - \frac{1}{2a}\right) , \tag{50}$$

and the corresponding potential energy by

$$\text{P.E.} = -\left(\frac{\mu}{r} + U\right) ; \tag{51}$$

hence $\mu/2a + U$ is an absolute constant. It follows that if a' is defined by

$$\frac{1}{a'} = \frac{1}{a}\left(1 + \frac{2aU}{\mu}\right) , \tag{52}$$

then a' is a totally invariant quantity to which the osculating semi-major axis approximates. As such, it is the obvious candidate[6] for the mean a that we shall need to adopt, and it is very gratifying that this choice ($\bar{a} = a'$) is demanded by other considerations.

Since a' is an absolute constant, so is n' , defined by

$$n'^2 a'^3 = \mu , \tag{53}$$

and we will find this of great importance[6] in the analysis of M , since it provides a firm point of reference for the perturbation in mean motion - the integration of δn has already been discussed in the derivation of (40).

3 MEAN ELEMENTS AND COORDINATES

3.1 Mean elements

Let ζ symbolize the generic osculating element, standing (in particular) for $a, e, i, \Omega, \omega, \sigma$ or M . The variation of M is given by (4), but the variation of the other ζ is (we are assuming) entirely due to J_2 and J_3 . Each variation is made up of a series of *short-period* terms that are essentially trigonometric functions of v , together with a long-term effect that is independent of v (mod 2π) and the orbital period. The combination of all the short-period terms in ζ constitutes the short-period perturbation, $\delta\zeta$,

removal of which from ζ gives a *mean* element, $\bar{\zeta}$, which for most purposes (because its variation can be plotted easily and accurately over long periods of time) is more useful than the osculating element; thus

$$\zeta = \bar{\zeta} + \delta\zeta . \tag{54}$$

The variation of $\bar{\zeta}$ with time is given, of course, by

$$\dot{\bar{\zeta}} = \frac{d\bar{\zeta}}{dt} , \tag{55}$$

and $\dot{\zeta}$ and $\bar{\zeta}$ in general have two components: one, a purely *secular* component that only arises with some of the elements; the other, a *long-period* component induced by the secular perturbation in ω. This secular perturbation being a first-order effect of J_2 (and any other 'even' zonal harmonic that might be considered), long-period variation does not arise until the second order in pure J_2 analysis, and is a cross-coupling effect in J_2/J_3 analysis; we shall find that there are second-order long-period perturbations for every element but semi-major axis. We summarize the division of $\dot{\zeta}$ into its components by writing

$$\dot{\bar{\zeta}} = \dot{\bar{\zeta}}_{sec} + \dot{\bar{\zeta}}_{\ell p} , \tag{56}$$

where $\dot{\zeta}_{sec}$ can only have an unperturbed component when ζ is M.

Now $\dot{\zeta}_{sec}$ is constant, by definition of 'secular', so its integral, from epoch to current time, is simply $\dot{\zeta}_{sec}\, t$; also, the integral of $\dot{\zeta}_{\ell p}$ may conveniently be denoted by $\Delta\zeta$ (Ref 4 used $\delta\zeta_{\ell p}$). Thus the variation of $\bar{\zeta}$ from epoch is given by

$$\bar{\zeta} = \bar{\zeta}_0 + \dot{\bar{\zeta}}_{sec}\, t + \Delta\zeta . \tag{57}$$

There is an important distinction in form between short-period perturbations and long-period perturbations, as defined here, and it can be seen at once from equations (54) and (57). Equation (54) indicates that $\delta\zeta$ is defined quite independently of epoch, and cannot in general be zero at epoch unless special epochs (such as ascending nodes) are chosen to suit the particular definition of $\delta\zeta$. Equation (57), on the other hand, indicates that $\Delta\zeta$ is automatically zero at epoch ($t = 0$); $\Delta\zeta$ is, in fact, the *definite* integral of $\dot{\zeta}_{\ell p}$ from epoch to

t , whereas $\delta\zeta$ is essentially an *indefinite* integral. Many writers have worked with epoch-independent long-period perturbations, so that $\Delta\zeta$ behaves like $\delta\zeta$, but as the present author has remarked before[3], a heavy price has to be paid, in particular in the introduction of an intrinsically spurious singularity at the so-called critical inclinations (where $\dot{\bar{\omega}}_{sec}$ vanishes). Nothing is lost by having epoch-dependent $\Delta\zeta$, since this simply makes the long-period perturbations behave like the secular perturbations, $\dot{\zeta}_{sec}\,t$ being necessarily epoch-dependent.

Implicit in the last paragraph is the understanding that a short-period perturbation is only defined 'up to an arbitrary constant', and this is inherent in the original (54), in the concept of an indefinite integral, in the notion of epoch independence, and in the earlier reference to a 'particular definition of $\delta\zeta$ '. The choice of a particular constant in $\delta\zeta$, or the choice of a particular 'reference value' for $\bar{\zeta}$ (which amounts to the same thing), is a matter of considerable importance at the first-order level in J_2 , and it is unfortunately not always clear, from published papers, what constants the authors have chosen or the motivation for their choice. As was remarked before[3] (and more recently in Ref 6), there are several possible rationales. Thus, the constants may be chosen to make each $\delta\zeta$ unbiased in time, *ie* such that its value is zero when averaged with respect to t . This choice, which the use of the epithet 'mean' rather naturally suggests, leads to awkward constants (see equations (194) to (198) of Ref 3), so a better one is to make $\delta\zeta$ unbiased in respect of the transformed integration variable $\bar{v}$. The best strategy of all, however, is to make an overall choice of constants so as to get the simplest possible expressions in a particular set of quantities derived from the elements. This was the strategy of Refs 2 and 3, and the present Report continues the philosophy, the 'particular set of quantities' being the short-period perturbations in the system of spherical-polar coordinates to be introduced in section 3.3.

In Ref 3, explicit arbitrary constants were introduced into the expressions for $\delta\zeta$ derived from first-order J_2 analysis, so that the choices adopted by particular authors could be conveniently quoted and compared. This involved a generality that will not be repeated in the present paper, and in section 4 we use the 'best' constants from the start. In the present section we must briefly consider just one of these constants, the one associated with semi-major axis, and here the obvious (and best) choice recovers the energy constant a' introduced in section 2.3, this being sometimes referred to as the semi-major axis of Brouwer[13]. It was used at RAE until the advent of the program PROP[14-16], for which the decision was taken - mistakenly with hindsight - to follow the

Smithsonian Astrophysical Observatory by switching to the semi-major axis of Kozai[17].

The choice of $\bar{a}$ also bears upon the choice of $\bar{n}$, since this must at first sight be tied to $\bar{a}$ by a barred version of (1), in which case (53) demands that $\bar{n} = n'$. Kepler's third law does not *have* to be carried over in this unmodified way, however[6], and in spite of the vital role played by n', there is a simpler and more natural choice of $\bar{n}$. To arrive at this choice, we set ζ to M in the generic (57), getting

$$\bar{M} = \bar{M}_0 + \dot{\bar{M}}_{sec} t + \Delta M , \qquad (58)$$

where

$$\dot{\bar{M}}_{sec} = \dot{\bar{\sigma}}_{sec} + \dot{\int}_{sec} \qquad (59)$$

and

$$\Delta M = \Delta\sigma + \Delta\int . \qquad (60)$$

Here $\int$ is the shorthand quantity introduced by (3), and we can therefore write

$$\int = n't + \int_0^t (n - n')\, d\tau . \qquad (61)$$

From (1) and (53) we can relate $n - n'$ to δa ($= a - a'$) and then integrate using (40). (The notation δn has not been used for $n - n'$, since it more appropriately applies to $n - \bar{n}$.) The integral in (61) has a secular component, $(dn')t$ say, such that $n' + dn'$ constitutes $\dot{\int}_{sec}$. Then the natural definition of $\bar{n}$ is given by

$$\bar{n} = \dot{\bar{M}}_{sec} = \dot{\bar{\sigma}}_{sec} + n' + dn' . \qquad (62)$$

If $\bar{n}$ is chosen in this way, it becomes part of the task of orbit analysis to express Kepler's third law in an appropriately modified form[6] such that $\bar{n}$ can be correctly derived from $\bar{a}$.

We shall find, in section 4, that in the first-order J_2 analysis $\bar{n}$ is identical with n', so that the Kepler law is valid for mean elements without modification - in fact it is (53). In section 5, however, we shall find this to

be no longer true at the second-order level; $\bar{n}$ differs from n', and the modified Kepler law is given by (299).

It is natural, and convenient, to extend the concept of 'mean' from the standard orbital elements to other quantities, *eg* v and r. The basis for definition is that the expression for a mean quantity in terms of mean elements is the same as the expression for the original quantity in terms of osculating elements. Thus the equations of section 2.1 hold true after the addition of bars to all quantities, the only (possible) exception being (1) - for the reason that has just been given. As an example, (5) yields

$$\bar{p}/\bar{r} = 1 + \bar{e}\cos\bar{v}\,, \tag{63}$$

after which the quantity $\bar{W}$, anticipated at (39), becomes available. (Obviously, $\bar{v}$ and $\bar{r}$ are not like the $\bar{\zeta}$ in being free of short-periodic variation!)

3.2 Semi-mean elements

An important complication arises in regard to the derivation and use of the mean elements, $\bar{\zeta}$, and their smooth (free of short-period variation) rate of change, $\dot{\bar{\zeta}}$; it relates to the change of variable from t to $\bar{v}$ when integrating the planetary equations, (22)-(27). The integration of secular and long-period terms in $d\zeta/d\bar{v}$ leads to long-term variation that is 'smooth in $\bar{v}$' rather than 'smooth in t', whilst the integration of short-period terms leads to a pure Poisson series (*ie* to a combination of trigonometric terms with arguments linear in $\bar{v}$ and $\bar{\omega}$). We can think of this long-term variation as applying to a *semi-mean* element $\tilde{\zeta}$, rather than to the true *mean* element, the relation between the two being of the form

$$\tilde{\zeta} = \bar{\zeta} + \hat{\zeta}\bar{m}\,; \tag{64}$$

here $\bar{m}$ is the 'mean equation of the centre' defined in accord with (8), whilst $\hat{\zeta}$ is essentially of the form $\dot{\bar{\zeta}}/\bar{n}$ as we shall see. (The qualification 'essentially' is because the precise definition we adopt, as most useful in practice, involves only the secular component of $\dot{\bar{\zeta}}$, being given by equation (67), but this is irrelevant for the moment.) To justify (64), suppose that $\hat{\zeta}$ has been introduced to denote the long-term component of $d\zeta/d\bar{v}$, defined from the right-hand side of the planetary equation for ζ after the change of variable. This integrates (at least for a purely secular long-term variation) to $\tilde{\zeta} - \tilde{\zeta}_0$, given by $\hat{\zeta}(\bar{v} - \bar{v}_0)$. But this may be rewritten as

$\hat{\zeta}(\bar{M} - \bar{M}_0) + \hat{\zeta}(\bar{m} - \bar{m}_0)$, where the first term is now smooth in t and is to be identified with $\dot{\bar{\zeta}}t$. This term constitutes $\bar{\zeta} - \bar{\zeta}_0$, so we have derived (64) in the form specifying variation from t_0 to t; the second term may be thought of as an induced 'carry-over' term.

It follows from (54) and (64) that

$$\zeta - \bar{\zeta} = \delta\zeta - \hat{\zeta}\bar{m} , \tag{65}$$

where in principle (until $\hat{\zeta}$ has been precisely defined) (65) expresses the pure-Poisson component, $\delta_p\zeta$ say, of the short-period perturbation, to which $\hat{\zeta}\bar{m}$ must be added to give the complete $\delta\zeta$. Now we are concerned with a relation (operating to second order) of the form

$$\zeta = \bar{\zeta} + (\dot{\bar{\zeta}}/\bar{n})\bar{m} + \delta_p\zeta , \tag{66}$$

in which the right-hand side reduces to a two-term sum if each component of the middle (induced) term is combined with either $\bar{\zeta}$ (contributing to $\tilde{\zeta}$) or $\delta_p\zeta$. In first-order J_2 analysis, the only induced terms to arise are those associated with $\dot{\Omega}_{sec}$ and $\dot{\omega}_{sec}$ (because of the identity between n' and $\bar{n}$), and it is natural to incorporate them with $\bar{\Omega}$ and $\bar{\omega}$, respectively, leaving just the pure $\delta_p\zeta$ to be amalgamated in the form of perturbations in spherical coordinates. On proceeding to second order (as in section 5), we find this first-order policy to have been essential (and not merely 'natural'), at any rate for ω, since awkward carry-over terms can only be avoided if $\tilde{\omega}$, not $\bar{\omega}$, appears in the arguments of the first-order terms. It is then obviously sensible for $\tilde{\Omega}$ and $\tilde{\omega}$ to include the second-order terms induced by $\dot{\Omega}_{sec}$ and $\dot{\omega}_{sec}$, as well as the first-order ones, and it will be found in section 5 that there also has to be a second-order term in $\tilde{M}$, induced by $\dot{\sigma}_{sec} + dn'$ from (62); there is no effect from n' (the dominant term in (62)) because this integrates directly to $n't$, without any need to transform the variable from t to $\bar{v}$. The induced effects of the long-period variation could be treated in the same way (and would probably have to be if the analysis were being taken to third order), but as $\dot{\zeta}_{\ell p}$ arises for every element (except a) it is simplest if the entire long-period component of the middle term of (66) is combined with $\delta_p\zeta$ rather than $\bar{\zeta}$; the resulting quantity may be denoted by $\delta\zeta$, and it is the $\delta\zeta$ that are amalgamated into coordinate perturbations.

Thus the precise definition of $\hat{\zeta}$ is

$$\hat{\zeta} = \dot{\tilde{\zeta}}_{sec}/\bar{n} \ , \qquad (67)$$

except that $\dot{M}_{sec}$ must be replaced by $\bar{n} - n'$ when ζ is M. With this definition, we can rewrite (66) as

$$\zeta = \tilde{\zeta} + \delta\zeta \ . \qquad (68)$$

We also require the time derivative of (64). With m defined by (8), $\dot{\bar{v}}$ given by (39), $\dot{M}$ taken as $\bar{n}$, and $\hat{\zeta}$ taken as constant, this derivative is given at once in the form

$$\dot{\tilde{\zeta}} = \dot{\bar{\zeta}} + \hat{\zeta}\bar{n}(\bar{W} - 1) \ . \qquad (69)$$

Now $\hat{\zeta}$ is given by (67). Also, it is legitimate and convenient to replace $\bar{W}$ by $\tilde{W}$, since they are the same to $O(J_2)$ and $\hat{\zeta}$ is $O(J_2)$, so we can rewrite (69) as

$$\dot{\tilde{\zeta}} = \tilde{W}\dot{\tilde{\zeta}}_{sec} + \dot{\tilde{\zeta}}_{\ell p} \ , \qquad (70)$$

except that (70) needs the extra term $n'(1 - \tilde{W})$ when ζ is M.

Explicit results for the perturbations in the orbital elements, associated with J_2, $J_2{}^2$ and J_3 respectively, are derived in sections 4, 5 and 7 of this paper, and it is convenient to represent the three sets of expressions in terms of a compact notation involving the suffixes 1, 2 and 3. Thus, for the secular perturbations we write

$$\hat{\zeta} = \bar{K}\,\hat{\zeta}_1 + \bar{K}^2\,\hat{\zeta}_2 \ (+\ \bar{H}\,\hat{\zeta}_3) \qquad (71)$$

(all the $\hat{\zeta}_3$ in fact being zero); for the long-period perturbations we write (since the $\zeta_{1,\ell p}$ are all zero)

$$\dot{\bar{\zeta}}_{\ell p} = \bar{n}\left(\bar{K}^2 \zeta_{2,\ell p} + \bar{H}\,\zeta_{3,\ell p}\right) ; \tag{72}$$

and for the pure short-period perturbations we write

$$\delta_p \zeta = \bar{K}\,\zeta_1 + \bar{K}^2\,\zeta_2 + \bar{H}\,\zeta_3 \ . \tag{73}$$

It then follows that

$$\delta\zeta = \bar{K}\,\zeta_1 + \bar{K}^2(\zeta_2 + \zeta_{2,\ell p}\,\bar{m}) + \bar{H}(\zeta_3 + \zeta_{3,\ell p}\bar{m}) \ , \tag{74}$$

it being immaterial whether we write $\bar{m}$ or $\tilde{m}$ in (74).

3.3 Satellite position and velocity

Given a set of mean orbital elements at epoch, the unimaginative and laborious way of computing the satellite's position and velocity at time t is to incorporate the appropriate perturbations, long-term and short-period, in each element, and then determine position and velocity from the resulting *osculating* elements. The algorithm for determining position from osculating elements was given before[2,3] (see also Ref 7), but for completeness it is given again here, with the necessary extension for generation of velocity, after which we return to the use of *mean* elements. The standard system of geocentric coordinates (x, y, z), is assumed, with x measured towards the vernal equinox and z towards the north pole. Then the algorithm for position and velocity from osculating elements (ζ) is as follows:

(i) the eccentric anomaly, E, is found by solving Kepler's equation[18,19]

$$E - e \sin E = M \ ; \tag{75}$$

(ii) v is found from one of the two equivalent formulae (apart from the ambiguity of quadrant in the first formula, which is automatically resolved if the Fortran ATAN2 function is used)

$$\tan v = \frac{q \sin E}{\cos E - e} \tag{76}$$

and

$$\tan \tfrac{1}{2}v = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \tan \tfrac{1}{2}E \ ; \tag{77}$$

(iii) u is obtained from (7), and r from either (5) or the equivalent formula

$$r = a(1 - e \cos E) \; ; \qquad (78)$$

(iv) n is obtained from (1), and then (38) gives $\dot{v}$, with which we can identify $\dot{u}$ (the dot notation is legitimate, because we are referring to a frozen osculating orbit, for which there are no perturbations, so that $dv/dt = \partial v/\partial t$ and $du/dt = \partial u/\partial t$);

(v) $\dot{r}$ is obtained from one of the equivalent formulae

$$\dot{r} = naeq^{-1} \sin v \qquad (79)$$

and

$$\dot{r} = ne(a^2/r) \sin E \; ; \qquad (80)$$

(vi) x, y and z are obtained from the double coordinate transformation expressed by the matrix formula

$$(x \quad y \quad z)^T = R_3(-\Omega)\, R_1(-i)\, (r \cos u \quad r \sin u \quad 0)^T \, , \qquad (81)$$

where $R_j(\;)$ describes rotation about the j^{th} axis, so that

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \text{ etc} \qquad (82)$$

and T in (81) denotes transposition;

(vii) $\dot{x}$, $\dot{y}$ and $\dot{z}$ are obtained by ∂-differentiation of (81), which gives

$$(\dot{x} \quad \dot{y} \quad \dot{z})^T = R_3(-\Omega)\, R_1(-i)\, (\dot{r} \cos u - r\dot{u} \sin u \quad \dot{r} \sin u + r\dot{u} \cos u \quad 0)^T \, . \qquad \ldots\ldots (83)$$

If M_0, the conceptual value of the mean anomaly at epoch (though defined at time t) is the starting point, rather than the mean anomaly at t itself, then there is of course a preliminary step given by $M = M_0 + nt$.

When we start from a set of $\bar{\zeta}_0$ (as opposed to ζ_0) and apply (57), followed by (54), to obtain the ζ from which the foregoing algorithm can be applied, then much the most involved part of the procedure arises with the computation of the $\delta\zeta$ to substitute in (54) (or in practice the $\delta\zeta$ to substitute in (68)). Hence the overall procedure will be greatly simplified if we can operate the position-velocity algorithm in a modified form, starting (at time t) from the $\bar{\zeta}$ (or alternatively the $\tilde{\zeta}$) instead of the ζ . The modified algorithm would then lead to '(semi-)mean position and velocity', after which it would (hopefully) be possible to add greatly simplified combinations of the $\delta\zeta$ directly to the position and velocity components.

It was in pursuance of this philosophy that Kozai[17] obtained first-order short-period perturbations, δr and δu , associated with J_2 , in a much more compact form than the perturbations, δa, δe, $\delta\omega$ and δM , that they displaced. These perturbations (δr and δu) were, in effect, the perturbations in a pair of two-dimensional polar coordinates (r and u), the plane of the coordinates being simply the osculating plane of the orbit. Much more recently, it has been pointed out by the present author[2,3] that it is logical to complete Kozai's approach by making the coordinate system three-dimensional, rotating but with a standard geocentric origin, with a reference plane that is now the 'mean orbital plane' instantaneously defined by $\bar{i}$ and $\bar{\Omega}$ rather than i and Ω . (It is the 'semi-mean orbital plane', based on $\tilde{\Omega}$ rather than $\bar{\Omega}$, that we strictly require, but this may be regarded as an unnecessary complication at this point.) On this basis we can interpret a set of mean orbital elements at time t as defining the position of a 'mean satellite', having mean anomaly $\bar{M}$ in a 'mean orbit' defined (instantaneously) by $\bar{a}$, $\bar{e}$ and $\bar{\omega}$; the mean orbit lies in (and moves with) the mean orbital plane. This is a helpful point of view for geometric visualization, but care is needed. Thus the mean satellite itself has a set of evolving osculating elements, and these cannot be identified with the mean elements of the actual satellite, in particular because the osculating elements of the mean satellite are not (in general) free of short-period variation - the underlying explanation of this is that the instantaneous mean elements only define the position, not the velocity, of the mean satellite*.

* For satellite motion in a uniformly rotating Keplerian ellipse in a uniformly rotating orbital plane (non-conservative example, in general), mean elements can be defined such that the 'mean satellite' actually coincides with the true satellite; then $\bar{i}$ identifies with i (osculating) at the 'apexes', but not with i at the 'nodes'. (This example underlies the choice of $\bar{i}$ in first-order J_2 analysis - see section 1.)

Refs 2 and 3 described a completion of the Kozai approach in which the axis system is referenced by cylindrical polar coordinates, the coordinates within the mean orbital plane (corresponding to r and u) being designated r' and u' , and the out-of-plane coordinate c . It would have been just as satisfactory to have used *spherical* polar coordinates, however, and in the complete (untruncated) analysis presented here it turns out that the spherical system is superior. The radial coordinate of the (actual) satellite in this system then naturally reverts to r , and the other two coordinates will be denoted by b and w , the (vector) relationship of r, b and w to rectangular coordinates, X, Y and Z , based on the mean orbital plane as XY-plane (with X towards the node), being

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} r \cos b \cos w \\ r \cos b \sin w \\ r \sin b \end{pmatrix} . \tag{84}$$

The directions of the axes of X and Y are not well defined for a near-equatorial mean orbital plane, but it will be found in due course that (so far as *position* is concerned) this causes no difficulty. The relationship of the new spherical system (r, b, w) to the (obsolete) cylindrical system is immediate, since in the old system we have $(X, Y, Z) = (r' \cos u', r' \sin u', c)$, but in view of the close relationship between the out-of-plane coordinates b and c (since $\sin b = c/r$) it is worth an explicit observation that, whereas c was the *third* cylindrical coordinate, b is the *second* spherical one. (Clearly, also $w = u'$ and $r^2 = r'^2 + c^2$.)

The position of the mean satellite is conceptually given by coordinates $\bar{r}$, $\bar{b}$ and $\bar{w}$, but it is really the 'semi-mean satellite' we are concerned with, so the reference point for (short-period) perturbations must actually be expressed as $(\tilde{r}, \tilde{b}, \tilde{w})$. Moreover, since Z is zero for the semi-mean satellite, we have from (84) that

$$\tilde{b} = 0 \tag{85}$$

and it is also clear that

$$\tilde{w} = \tilde{u} . \tag{86}$$

Since the whole point of this approach is that we apply (68) to coordinates, rather than elements, the spherical coordinates of the *actual* satellite are given by

$$(r,\ b,\ w) \;=\; (\tilde{r},\ \tilde{b},\ \tilde{w}) + (\delta r,\ \delta b,\ \delta w)\ , \tag{87}$$

with a natural extension of the notation that (68) introduced. From (87) we also have

$$(\dot{r},\ \dot{b},\ \dot{w}) \;=\; (\dot{\tilde{r}},\ \dot{\tilde{b}},\ \dot{\tilde{w}}) + (\delta\dot{r},\ \delta\dot{b},\ \delta\dot{w})\ , \tag{88}$$

on the understanding that $\delta\dot{r}$ (*etc*) denotes $d(\delta r)/dt$, rather than $\delta(dr/dt)$; the distinction arises because (in particular) $\dot{\tilde{r}}$ is not identical with $\tilde{\dot{r}}$, assuming the latter to be defined by converting the right-hand side of (80) to semi-mean elements.

We can now focus on the specific objective of this section, namely, the derivation of position and velocity at time t from mean* elements at epoch without any reference to osculating elements. The derivation of mean elements at t is necessary as a preliminary step, carried out in principle by application of (57); the practical difficulties in this step are considered in section 3.5 (and further in Part 3), so we may assume here a starting point consisting of the six $\bar{\zeta}$ at t. We confine ourselves initially to the derivation of position, and start by applying steps (i) and (ii) of the algorithm that was given for osculating elements; the result is the 'mean true anomaly' $\bar{v}$. Since $\bar{m}$ is now available, (64) gives the appropriate $\tilde{\zeta}$, *viz* $\tilde{\Omega}$, $\tilde{\omega}$ and $\tilde{M}$; then a repeat of steps (i) and (ii) (only done to reflect the change from $\bar{M}$ to $\tilde{M}$), followed by (iii), leads to $\tilde{v}$, $\tilde{r}$ and $\tilde{u}$. At this point, in view of (85) and (86), we have the semi-mean satellite position $(\tilde{r},\ \tilde{b},\ \tilde{w})$; on the assumption that formulae for δr, δb and δw are available, it follows that (87) yields (r, b, w), and then (84) gives (X, Y, Z). We finally require the formula corresponding to (81) in step (vi) of the algorithm, and thi may be written

$$(x \quad y \quad z)^T \;=\; R_3(-\tilde{\Omega})\, R_1(-\bar{i})\ \ (X \quad Y \quad Z)^T\ , \tag{89}$$

since there is no distinction between $\bar{i}$ and $\tilde{i}$.

It is a straightforward matter to express the generic first-order formulae for δr, δb and δw in terms of the $\delta\zeta$, and we now do this in advance of the specific first-order J_2 analysis of section 4 and the J_3 analysis of section 7.

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* To avoid confusion, it should be noted that the basic (reference) elements are mean, not semi-mean; in application of the 'position-velocity algorithm', however, the first requirement is that the $\bar{\zeta}$ be superseded by the $\tilde{\zeta}$, and it is the latter that define the spherical-coordinate system.

(Second-order formulae are much harder to express, however, and their derivation is deferred until they are needed, in section 6.) Our starting point is the identification of the two formulae for $(x \quad y \quad z)^T$, *viz* (81) and (89). Using also (84) in the identification, we get

$$\begin{pmatrix} \cos b \cos w \\ \cos b \sin w \\ \sin b \end{pmatrix} = R_1(\bar{i})\ R_3(\tilde{\Omega})\ R_3(-\Omega)\ R_1(-i) \begin{pmatrix} \cos u \\ \sin u \\ 0 \end{pmatrix} . \tag{90}$$

Now

$$R_3(\tilde{\Omega})\ R_3(-\Omega) = R_3(-\delta\Omega) , \tag{91}$$

exactly, but first-order approximation is required to complete the reduction, which leads to

$$b = \sin u\ \delta i - s \cos u\ \delta\Omega \tag{92}$$

and

$$w = u + c\ \delta\Omega . \tag{93}$$

In view of (85) we can write δb in place of b in (92), and in view of (86), which can be subtracted from (93), we also get

$$\delta w = \delta u + c\ \delta\Omega . \tag{94}$$

For the formulae we are seeking, it only remains to express δr and δu in terms of the $\delta\zeta$, and the required expressions are immediate from (41) and (42). We thus obtain (generically)

$$\delta r = (r/a)\ \delta a - (a \cos v)\ \delta e + (aeq^{-1} \sin v)\ \delta M , \tag{95}$$

$$\delta b = (\sin u)\ \delta i - (s \cos u)\ \delta\Omega \tag{96}$$

and

$$\delta w = \delta\omega + c\ \delta\Omega + q^{-2} \sin v\ (2 + e \cos v)\ \delta e + W\,\delta M . \tag{97}$$

In practice, of course, the coefficients in (95)-(97) are to be interpreted as semi-mean quantities and not as osculating ones.

It was remarked, earlier in the section, that no difficulty in *position* computation arises from the indeterminacy (for near-equatorial orbits) of the X-axis (origin for w) in the rotating system of spherical coordinates, and this needs to be justified before we pass on to *velocity* (for which there *is* a difficulty - to be dealt with in the next section). The underlying principle here is that a set of standard elements (osculating, mean or semi-mean) can always be safely used (for position computation), so long as their values are consistent. All this means is that no matter how badly defined the elements Ω, ω and M are individually, no accuracy is lost so long as they are such that their non-singular combinations (*cf* the quantities ψ and L of section 2) retain full accuracy.

Given a consistent set of $\bar{\zeta}_0$ (mean elements at epoch), the first step in the computation of satellite position at time t is, as already noted, the application of long-term perturbations, to produce $\bar{\zeta}$ at t. This must be done with full allowance for the possibility of singularity, as described in section 3.5, after which we have (at time t) a consistent set of mean elements and hence a consistent set of semi-mean elements. If $\tilde{\Omega}$ is not well defined in this set, then the X-axis in the semi-mean orbital plane must also be ill defined; but this will not matter, assuming that the value of $\tilde{\omega}$, to be added to $\tilde{v}$ to give $\tilde{u}$ ($= \tilde{w}$), is consistent with $\tilde{\Omega}$. In other words, the coordinates (X and Y) given by (84) will only be arbitrary to an extent that is precisely compensated for in the application of the correspondingly arbitrary $\tilde{\Omega}$ in (99). This is true even in the extreme case - which can arise in practice as will be seen in section 7.4 - when $\dot{\tilde{\Omega}}$ becomes infinite, but such an infinity inevitably has a more profound effect on *velocity*, which is why the velocity formulae to be presented at the end of this section are only provisional.

Thus a nominal indeterminacy for $\tilde{w}$ in $(\tilde{r}, \tilde{b}, \tilde{w})$ is dealt with easily enough. For the perturbations $(\delta r, \delta b, \delta w)$ the situation is even better, since these are well determined at all times. This is essentially because of the $\delta\Omega$ term in (94), which reflects the fact that δw (unlike δu) is tied to the same reference direction as w itself (Kozai's δu is free of 'circular singularity' but not of 'equatorial singularity').

We now proceed to the formulae for $\dot{x}$, $\dot{y}$ and $\dot{z}$, to complete the algorithm for position and velocity, bearing in mind that the formulae are only provisional. The adjustments to avoid infinities are given in the next section.

We have to cover the three remaining steps - (iv), (v) and (vii) - in the adaptation of the original algorithm to the use of (semi-)mean elements. It

follows from the way in which $\tilde{r}$ is defined that we can invoke (41) again, as in the derivation of (95), to obtain $\dot{\tilde{r}}$; since $\dot{\bar{a}} = 0$ we get

$$\dot{\tilde{r}} = (\bar{a}\,\bar{e}\,\bar{q}^{-1}\sin\tilde{v})\,\dot{\tilde{M}} - (\bar{a}\cos\tilde{v})\,\dot{\bar{e}} \ . \tag{98}$$

Clearly $\dot{\tilde{b}}$ is zero. Finally, (42) can be invoked again, with (86), to give $\dot{\tilde{w}}$, the result being

$$\dot{\tilde{w}} = \dot{\tilde{\omega}} + \tilde{W}\dot{\tilde{M}} + \bar{q}^{-2}\sin\tilde{v}\,(2 + \bar{e}\cos\tilde{v})\,\dot{\bar{e}} \ . \tag{99}$$

The absence of a term in $\dot{\tilde{\Omega}}$ (*cf* (97)) is due to the point just made in relation to δw and w ; $\dot{\tilde{\Omega}}$ is allowed for separately (as we are about to see) via the motion of the (semi-)mean orbital plane, whereas $\delta\Omega$ is resolved directly into contributions δb and δw .

From $(\dot{\tilde{r}}, \dot{\tilde{b}}, \dot{\tilde{w}})$ we obtain $(\dot{r}, \dot{b}, \dot{w})$, by appeal to (88), $(\delta\dot{r}, \delta\dot{b}, \delta\dot{w})$ being given by differentiation of $(\delta r, \delta b, \delta w)$. The resulting short-period expressions involve combinations of $\dot{\tilde{v}}$ and $\dot{\tilde{\omega}}$, including $\dot{\tilde{u}}\ (= \dot{\tilde{v}} + \dot{\tilde{\omega}})$, but first-order representations of these are adequate for our second-order analysis, so the singularities associated with long-period perturbations in Ω and ω cause no difficulties in these expressions.

It remains to obtain a formula having the same relation to (83) as (89) has to (81). Differentiating (89) directly, we get

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \left\{ R_3(-\tilde{\Omega})\ R_1'(-\bar{i})\ \dot{\bar{i}} + R_3'(-\tilde{\Omega})\ R_1(-\bar{i})\,\dot{\tilde{\Omega}} \right\} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + R_3(-\tilde{\Omega})\ R_1(-\bar{i}) \begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} , \qquad \ldots\ldots \tag{100}$$

where $$R_1'(-\theta) = dR_1(-\theta)/d\theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{pmatrix} (etc) \ . \tag{101}$$

Now from (84) we have

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} \cos b \cos w & -\sin b \cos w & -\sin w \\ \cos b \sin w & -\sin b \sin w & \cos w \\ \sin b & \cos b & 0 \end{pmatrix} \begin{pmatrix} \dot{r} \\ r\dot{b} \\ r\dot{w}\cos b \end{pmatrix} . \tag{102}$$

Also, since $\dot{\hat{i}}_{soc} = 0$, we have

$$\dot{\hat{i}} = \dot{\hat{i}}_{\ell p} , \tag{103}$$

whilst

$$\dot{\hat{\Omega}} = \dot{\hat{\Omega}}_{\ell p} + \tilde{W}\,\dot{\hat{\Omega}}_{soc} \tag{104}$$

by (70). It may be observed that the coefficient of $\dot{\hat{i}}$ in (100) can be expressed as $(z \sin \tilde{\Omega} \quad - z \cos \tilde{\Omega} \quad \bar{c}Y - \bar{s}Z)^T$, whilst the coefficient of $\tilde{\Omega}$ can be expressed as $(-y \quad x \quad 0)^T$.

3.4 Avoidance of singularity in velocity computation

The formulae at the end of the last section being unsuitable for universal velocity computation as they stand, because of possible singularity, we must start by identifying the potentially infinite quantities. They are $\dot{\hat{\omega}}$ and $\dot{\hat{M}}$ in (99), which propagate into $\dot{w}$ in (102), and the compensating $\tilde{\Omega}$ in (100).

The quantity $\dot{\hat{M}}$ is potentially singular through expressions for $\dot{\hat{M}}_{\ell p}$ that contain $\bar{e}^{-1}$ as a factor - see (399) in particular. This causes no problem in (98), where $\dot{\hat{M}}$ occurs with a multiplying $\bar{e}$, but in (99) the potential infinity is real. It is convenient to deal with the two formulae in the same way, however, and we seek to replace $\dot{\hat{M}}$ by as close a non-singular equivalent as possible.

From the special case of (70) when ζ is M ,

$$\dot{\hat{M}} = n' + \tilde{W}(\bar{n} - n') + \dot{\hat{M}}_{\ell p} , \tag{105}$$

with only the last term potentially singular; this can also be written as

$$\dot{\hat{M}} = \bar{n} + (\tilde{W} - 1)(\bar{n} - n') + \dot{\hat{M}}_{\ell p} . \tag{106}$$

We now recall that $\dot{L}$, given by (35), cannot be singular, so we introduce a new quantity, N (designated $\bar{n}_L$ in Ref 4), that can be regarded as a modified version of the mean motion (though it is not free of short-period variation, let alone constant); it is defined by

$$N = \bar{n} + (\tilde{W} - 1)(\bar{n} - n') + \dot{L}_{\ell p} . \tag{107}$$

Then from (106) and (35) we have

$$\dot{M} = N - \bar{q}\,\dot{\Psi}_{\ell p} . \tag{108}$$

Hence, since $\dot{e}_{sec} = 0$, (98) can be rewritten as

$$\dot{r} = (\bar{a}\,\bar{e}\,\bar{q}^{-1} \sin \tilde{v})\,N - (\bar{a} \cos \tilde{v})\,\dot{e}_{\ell p} - (\bar{a}\,\bar{e} \sin \tilde{v})\,\dot{\Psi}_{\ell p} . \tag{109}$$

All terms here are free of singularity, the last term (like the original $\dot{M}$ term) being so because of the $\bar{e}$ factor.

Our real concern is to rewrite (99), and this introduces the complication that (as we have seen) the quantity on the left-hand side, *viz* $\dot{w}$, itself has the potential singularity, so that a modified quantity must be expressed. From (70), with ζ taken as ω, and (108), we have

$$\dot{\omega} + \tilde{W}\,\dot{M} = \tilde{W}\left(N + \dot{\omega}_{sec} - \bar{q}\,\dot{\Psi}_{\ell p}\right) + \dot{\omega}_{\ell p} , \tag{110}$$

in which the last term is potentially singular for (both) circular and equatorial orbits, but even the term in $\dot{\Psi}_{\ell p}$ is no longer singularity-free. However, by addition of $\bar{c}\,\dot{\Omega}_{\ell p}$ to both sides of (110), we may write

$$\dot{\omega} + \tilde{W}\,\dot{M} + \bar{c}\,\dot{\Omega}_{\ell p} = \tilde{W}\left(N + \dot{\omega}_{sec}\right) - (\bar{q}\,\tilde{W} - 1)\,\dot{\Psi}_{\ell p} , \tag{111}$$

and now the right-hand side is well behaved, since

$$\bar{q}\,\tilde{W} - 1 = \bar{e}\,\bar{q}^{-2}\left\{2 \cos \tilde{v} + \bar{e}(1 + \cos^2 \tilde{v})\right\} . \tag{112}$$

It immediately follows that the modified (non-singular) version of (99) that we require is

$$\dot{\tilde{w}}' = \tilde{W}(N + \dot{\bar{\omega}}_{sec}) + \bar{q}^{-2}\Big[\sin\tilde{v}\,(2 + \bar{e}\cos\tilde{v})\,\hat{\dot{e}}_{\ell p}$$
$$- \bar{e}\left\{2\cos\tilde{v} + \bar{e}\,(1+\cos^2\tilde{v})\right\}\hat{\dot{\Psi}}_{\ell p}\Big], \qquad \text{......(113)}$$

where $\dot{\tilde{w}}'$ is defined by

$$\dot{\tilde{w}}' = \dot{\tilde{w}} + \bar{c}\,\hat{\dot{\Omega}}_{\ell p}\,. \qquad (114)$$

(It is remarked that $\dot{\tilde{w}}'$ is the appropriate quantity to use as the derivative of $\tilde{u}$, when differentiating the trigonometric arguments of the individual terms of δr, δb and δw to provide $\delta\dot{r}$, $\delta\dot{b}$ and $\delta\dot{w}$; (86) suggests it is $\dot{\tilde{w}}$ rather than $\dot{\tilde{w}}'$ that is required, but $\dot{\tilde{w}}$ is not singularity-free; also, the difference between $\dot{\tilde{w}}$ and $\dot{\tilde{w}}'$ can be neglected in $\delta\dot{r}$ *etc*, since the resulting effects are $O(J_2^3, J_2J_3)$.)

Since $\dot{\tilde{w}}'$, rather than $\dot{\tilde{w}}$, is the non-singular quantity, it follows that $\dot{X}$ and $\dot{Y}$, as given by (102), should be replaced by quantities computed from $\dot{w}'$ rather than $\dot{w}$, where $\dot{w}'$ is obtained from addition of $\delta\dot{w}$ to $\dot{\tilde{w}}'$ instead of to $\dot{\tilde{w}}$. This will make the last term of (100) non-singular and, as we shall see, in such a way as to make the middle term non-singular as well. What we are doing to (102) amounts to the creation of errors $-r\,\bar{c}\,\hat{\dot{\Omega}}_{\ell p}\cos b\sin w$ and $r\,\bar{c}\,\hat{\dot{\Omega}}_{\ell p}\cos b\cos w$, in $\dot{X}$ and $\dot{Y}$ respectively, that can be corrected in (100) by the additional term

$$r\,\bar{c}\,\hat{\dot{\Omega}}_{\ell p}\begin{pmatrix}\cos\tilde{\Omega} & -\bar{c}\sin\tilde{\Omega} & \bar{s}\sin\tilde{\Omega}\\ \sin\tilde{\Omega} & \bar{c}\cos\tilde{\Omega} & -\bar{s}\cos\tilde{\Omega}\\ 0 & \bar{s} & \bar{c}\end{pmatrix}\begin{pmatrix}\cos b\sin w\\ -\cos b\cos w\\ 0\end{pmatrix},$$

on expanding $R_3(-\tilde{\Omega})\,R_1(-\bar{i})$, the multiplier of $(\dot{X}\ \ \dot{Y}\ \ \dot{Z})^T$. But the additional term can be rewritten as

$$r\,\bar{c}\,\dot{\Omega}_{\ell p}\begin{pmatrix}\bar{c}\sin\tilde{\Omega} & \cos\tilde{\Omega} & 0\\ -\bar{c}\cos\tilde{\Omega} & \sin\tilde{\Omega} & 0\\ -\bar{s} & 0 & 0\end{pmatrix}\begin{pmatrix}\cos b\cos w\\ \cos b\sin w\\ \sin b\end{pmatrix},$$

which is in just the right form for combination with the $\dot{\Omega}_{\ell p}$ component of the middle term of (100), which can itself be written as

$$r\,\dot{\Omega}_{\ell p}\begin{pmatrix}-\sin\tilde{\Omega} & -\bar{c}\cos\tilde{\Omega} & \bar{s}\cos\tilde{\Omega}\\ \cos\tilde{\Omega} & -\bar{c}\sin\tilde{\Omega} & \bar{s}\sin\tilde{\Omega}\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\cos b\cos w\\ \cos b\sin w\\ \sin b\end{pmatrix}.$$

On combination, and in view of (84), we get

$$\bar{s}\,\dot{\Omega}_{\ell p}\begin{pmatrix}-\bar{s}\sin\tilde{\Omega} & 0 & \cos\tilde{\Omega}\\ \bar{s}\cos\tilde{\Omega} & 0 & \sin\tilde{\Omega}\\ -\bar{c} & 0 & 0\end{pmatrix}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix}.$$

This can be rewritten as $\bar{s}\,\dot{\Omega}_{\ell p}\,R_3(-\tilde{\Omega})\,R_1(-\bar{i})\,(Z\quad 0\quad -X)^T$; the appearance of $\bar{s}$, effectively replacing $\bar{c}$ as a multiplier of $\dot{\Omega}_{\ell p}$, makes the expression fully non-singular, and its form is such that it can be gathered back into the last term of (100).

Only the secular component of $\dot{\tilde{\Omega}}$ in (100) now remains and it follows at once, in view of the simplifications indicated by the last sentence of section 3.3, that the equation can at last be rewritten, using (104), as

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} = \dot{\bar{i}}\begin{pmatrix}z\sin\tilde{\Omega}\\ -z\cos\tilde{\Omega}\\ \bar{c}Y-\bar{s}Z\end{pmatrix} + \tilde{W}\,\dot{\tilde{\Omega}}_{sec}\begin{pmatrix}-y\\ x\\ 0\end{pmatrix} + R_3(-\tilde{\Omega})\,R_1(-\bar{i})\begin{pmatrix}\dot{X}'\\ \dot{Y}'\\ \dot{Z}'\end{pmatrix}, \qquad (115)$$

where $\dot{X}'$, $\dot{Y}'$ and $\dot{Z}'$ are non-singular versions of $\dot{X}$, $\dot{Y}$ and $\dot{Z}$, given by

$$\dot{X}' = \dot{r}\cos b\cos w - r\sin b\,(\dot{b}\cos w - \bar{s}\dot{\tilde{\Omega}}_{\ell p}) - r\dot{w}'\cos b\sin w, \qquad (116)$$

$$\dot{Y}' = \dot{r}\cos b\sin w - r\dot{b}\sin b\sin w + r\dot{w}'\cos b\cos w \qquad (117)$$

and

$$\dot{z}' = \dot{r} \sin b + r \cos b \, (\dot{b} - \bar{s}\dot{\bar{\Omega}}_{\ell p} \cos w) \; ; \qquad (118)$$

here $\dot{w}'$, as indicated after (114), is given by

$$\dot{w}' = \dot{\bar{w}} + \delta\dot{w} + \bar{c}\dot{\bar{\Omega}}_{\ell p} \; . \qquad (119)$$

3.5 Long-term perturbations and singularity

We have seen how to get from the mean elements, $\bar{\zeta}$, at time t , to the components of position and velocity in the normal (equatorial) axis system, without any real problem associated with singularity. It remains to consider such difficulties as exist in the propagation of the mean elements themselves, from their values $(\bar{\zeta}_0)$ at epoch, this having been described in section 3.3 as 'a preliminary step'. The schematic propagation formula is (57), and the only difficulties turn out to be with the long-period terms, $\Delta\zeta$. One difficulty relates to the general method of evaluating each $\Delta\zeta$ as a well-behaved integral of $\dot{\zeta}_{\ell p}$, and the others relate to the way in which the circular and equatorial singularities are dealt with. We give a general consideration here of how the difficulties can be overcome, reserving the full details, for $O(J_2^2, J_3)$ perturbations, to Part 3.

(a) General method of evaluating the $\Delta\zeta$

As remarked in section 3.1, the $\Delta\zeta$ arise, by definition, from components of $\dot{\zeta}$ in which ω , but not v , is an explicit argument, the point here being that ω has a first-order secular variation due to J_2 . For the general zonal harmonic, J_ℓ , there are trigonometric terms in $k\omega$ for all values of k , up to $\ell - 2$, having the same parity as ℓ , so that for J_3 we just have (as we shall see in section 7) terms in $\cos\omega$ and $\sin\omega$. If the elements are to be propagated over long periods of time, it is obviously desirable that the J_3 analysis should reflect the secular variation of ω in the integration of these terms, even when the analysis is only taken to the first order in J_3 . (The question of the precise meaning of the terms 'first-order', 'second-order', etc, was considered in section 9 of Ref 3 and that discussion will not be repeated in detail here.) On taking J_2 analysis to second order, we have similarly (as we shall see in section 5) terms in $\cos 2\omega$ and $\sin 2\omega$, integration of which should again allow for the first-order variation in ω . On this basis (and

with J_3 regarded as second-order in J_2) we derive the complete formal* second-order solution that we require; it will be found that we need to incorporate some terms that (as Ref 3 argues) may formally be regarded as part of the third-order variation.

To specify the problem precisely, we write

$$C_k = \int_0^t \cos k\bar{\bar{\omega}}\, d\tau \quad \text{and} \quad S_k = \int_0^t \sin k\bar{\bar{\omega}}\, d\tau , \tag{120}$$

where $\bar{\bar{\omega}}$, written $\bar{\bar{\omega}}_\tau$ to avoid ambiguity, is given by

$$\bar{\bar{\omega}}_\tau = \bar{\omega}_0 + \dot{\bar{\omega}}_{sec}\, \tau , \tag{121}$$

observing that $\bar{\bar{\omega}}$ is not the same as $\bar{\omega}$ (except at $\tau = 0$) , because it does not include the long-period term $\Delta\omega$. Then the problem is to have formulae for evaluation of C_k and S_k that are 'universally computable' in giving accurate values for all values of $\dot{\bar{\omega}}_{sec}$; this includes (in particular) zero, $\dot{\bar{\omega}}_{sec}$ being zero (as far as the first-order J_2 formulae are concerned) at the so-called critical inclinations given (as we shall see from (152)) by $g = 0$.

The formal evaluation of the definite integral C_k is, of course, given by $(\sin k\bar{\bar{\omega}}_t - \sin k\bar{\omega}_0)/k\dot{\bar{\omega}}_{sec}$, but this reduces to the indeterminate $0/0$, rather than the determinate $t \cos k\bar{\omega}_0$, as $\dot{\bar{\omega}}_{sec}$ tends to zero. To avoid this situation, we follow, with modifications, the course taken by Merson[14], who introduced the bounded functions F_1, F_2 and F_3 , given by

$$F_1(\theta) = \theta^{-1} \sin\theta , \tag{122}$$

$$F_2(\theta) = \theta^{-2} (1 - \cos\theta) \tag{123}$$

and

$$F_3(\theta) = \theta^{-3} (\theta - \sin\theta) . \tag{124}$$

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* The errors in the formally complete second-order solution can only be regarded as truly $o(J_2^2, J_3)$, *ie* $O(J_2^3, J_2J_3)$, if the timescale is sufficiently short, *ie* t must be $o(J_2^{-1})$; generally speaking, a formal solution of j'th order in J_2 will produce errors that are $O(J_2^{j+1}\, t)$ in the long-term variation over time t . This point was not made in Ref 3.

For values of θ that are 'far enough from zero' these functions can be computed, without loss of accuracy, from their defining expressions, whilst for values near zero they can be computed from a small number of terms of their power-series expansions, which are then rapidly convergent, the limiting values of $F_1(\theta)$, $F_2(\theta)$ and $F_3(\theta)$ being 1, $\frac{1}{2}$ and $\frac{1}{6}$ respectively. (Merson's functions are equivalent to three of the functions introduced by Stumpff[20].) We also define, for convenience, $\nabla\omega$ given by

$$\nabla\omega = \dot{\bar{\omega}}_{sec}\, t \,. \tag{125}$$

Merson's formula for C_k was derived on the basis that

$$\begin{aligned}\sin k\bar{\bar{\omega}}_t - \sin k\bar{\omega}_0 &= \sin k(\bar{\omega}_0 + \nabla\omega) - \sin k\bar{\omega}_0 \\ &= \sin k\nabla\omega \cos k\bar{\omega}_0 - (1 - \cos k\nabla\omega) \sin k\bar{\omega}_0 \,,\end{aligned} \tag{126}$$

so that he obtained (with different notation)

$$C_k = \left\{F_1(k\nabla\omega) \cos k\bar{\omega}_0 - k\nabla\omega\, F_2(k\nabla\omega) \sin k\bar{\omega}_0\right\} t \,. \tag{127}$$

However, it is more natural to write

$$\begin{aligned}\sin k\bar{\bar{\omega}}_t - \sin k\bar{\omega}_0 &= 2 \sin \tfrac{1}{2}k(\bar{\bar{\omega}}_t - \bar{\omega}_0) \cos \tfrac{1}{2}k(\bar{\omega}_0 + \bar{\bar{\omega}}_t) \\ &= 2 \sin \tfrac{1}{2}k\nabla\omega \cos k\bar{\bar{\omega}}_{\frac{1}{2}t} \,,\end{aligned} \tag{128}$$

since we then have the simpler expression

$$C_k = \left\{F_1(\tfrac{1}{2}k\nabla\omega) \cos k\bar{\bar{\omega}}_{\frac{1}{2}t}\right\} t \,; \tag{129}$$

the corresponding expression for S_k is

$$S_k = \left\{F_1(\tfrac{1}{2}k\nabla\omega) \sin k\bar{\bar{\omega}}_{\frac{1}{2}t}\right\} t \,. \tag{130}$$

Use of (129) and (130) removes all difficulty associated with the general evaluation of the $\Delta\zeta$, except that for each of $\Delta\Omega$ and $\Delta\omega$ it is necessary to allow for a further term that is important though formally only of third order[3]. These terms are induced in the first-order effects of $\dot{\Omega}_{sec}$ and $\dot{\omega}_{sec}$ by the second-order Δe and Δi , since $\dot{\Omega}_{sec}$ and $\dot{\omega}_{sec}$ are functions of $\bar{e}$ and $\bar{i}$. The computation of these terms involves the integration of Δe and Δi , so we require universally computable formulae for the (definite) integrals of C_k and S_k - specifically of S_2 for $J_2{}^2$ analysis and C_1 for J_3 analysis.

The formal expression for the integral, from $\tau = 0$ to $\tau = t$, of C_k (as a function of τ) is $-(\cos k\bar{\bar{\omega}}_t - \cos k\bar{\bar{\omega}}_0)/(k\dot{\omega}_{sec})^2 - (\sin k\bar{\bar{\omega}}_0 / k\dot{\omega}_{sec})t$. We can write this as

$$\frac{2\sin \tfrac{1}{2}k\nabla\omega \sin k\bar{\bar{\omega}}_{\frac{1}{2}t}}{(k\dot{\omega}_{sec})^2} + \frac{2\sin \tfrac{1}{4}k\nabla\omega \cos k\bar{\bar{\omega}}_{\frac{1}{4}t} - \sin k\bar{\bar{\omega}}_{\frac{1}{2}t}}{k\dot{\omega}_{sec}}\, t \;,$$

and then rearrange the terms to give

$$\frac{2\sin \tfrac{1}{4}k\nabla\omega \cos k\bar{\bar{\omega}}_{\frac{1}{4}t}}{k\dot{\omega}_{sec}}\, t - \frac{\sin k\bar{\bar{\omega}}_{\frac{1}{2}t}}{(k\dot{\omega}_{sec})^2}\left(k\dot{\omega}_{sec}\, t - 2\sin \tfrac{1}{2}k\nabla\omega\right) \;.$$

From this we obtain at once the required result, *viz*

$$\int_0^t C_k \, d\tau = \tfrac{1}{2}\left[F_1(\tfrac{1}{4}k\nabla\omega)\cos k\bar{\bar{\omega}}_{\frac{1}{4}t} - \tfrac{1}{2}k\nabla\omega\, F_3(\tfrac{1}{2}k\nabla\omega)\sin k\bar{\bar{\omega}}_{\frac{1}{2}t}\right] t^2 \;; \qquad (131)$$

similarly

$$\int_0^t S_k \, d\tau = \tfrac{1}{2}\left[F_1(\tfrac{1}{4}k\nabla\omega)\sin k\bar{\bar{\omega}}_{\frac{1}{4}t} + \tfrac{1}{2}k\nabla\omega\, F_3(\tfrac{1}{2}k\nabla\omega)\cos k\bar{\bar{\omega}}_{\frac{1}{2}t}\right] t^2 \;. \qquad (132)$$

When the induced contributions to $\Delta\Omega$ and $\Delta\omega$ are allowed for, the propagation of the $\bar{\zeta}$ should (apart from the question of singularity, still to

be discussed) give accurate results over quite long periods of time. As indicated by the last footnote, however, there will be an increasing need, in modelling the long-term variation as the number of revolutions from epoch builds up, for terms that are formally of the third order. Without going to these terms, accuracy can in principle be improved if a long interval is split into short intervals, as a form of 'rectification' involving subsidiary epochs. This brings in the hybrid (or semi-analytical) component of the orbit generator, referred to in section 1 and to be described in Part 3. Ephemerides, generated with and without rectification, will be compared in Part 3, and assessed against ephemerides produced by pure numerical integration; this assessment provides the ultimate justification for all the formulae developed in the present section of the present Report.

(b) Singularity avoidance for near-circular orbits

It will be found from equations (389) and (399) that $\omega_{3,\ell p}$ and $M_{3,\ell p}$, as defined by equation (72), are infinite when e is zero, this being a manifestation of the circular-orbit singularity that has been referred to several times. However, the phenomenon is merely a consequence of the use of elements that purport to define the position of perigee accurately (relative to the ascending node, in regard to ω, and relative to the satellite's (mean) position, in regard to M) in circumstances where the concept of 'perigee' ceases to be meaningful. If the elements e, ω and M are replaced by the quantities ξ and η, defined by (9) and (10), and U ($= M + \omega$), then the difficulty vanishes.

It is not a satisfactory procedure to work throughout with ξ, η and U as elements, however. For one thing the analysis becomes much more complicated, particularly (as we shall see) in respect of U. For another, ξ and η themselves become undefined when the orbit is equatorial but not circular, but we defer all questions of the equatorial singularities until sub-section (c). Finally, it is essential that the dominant (first-order) component of the secular perturbation, $\dot{\omega}_{sec}\,t$, be applied to $\tilde{\omega}$ directly, since the perturbation occurs precisely and naturally (and hence necessarily non-singularly) in this form. Fortunately, the effect of working throughout in terms of ξ, η and U can be obtained by appeal to their differential relationships to e, ω and M, without introducing them explicitly at all. Thus, since $\tilde{e}\,\Delta\omega$ is finite for all $\tilde{e}$, and may be thought of as a single entity to be operated with instead of $\Delta\omega$, we compute $\tilde{e}$ and $\tilde{\omega}$ (the mean values of e and ω at time t) via $\bar{\bar{e}}$ and $\bar{\bar{\omega}}$, where $\bar{\bar{\omega}}$ was introduced, at (121), to cover secular perturbations alone, and $\bar{\bar{e}}$ (defined on the same basis) is equal to $\bar{e}_0$; the formulae required are

$$\bar{e}\cos\bar{\omega} = (\bar{\bar{e}} + \Delta e)\cos\bar{\bar{\omega}} - (\bar{e}\,\Delta\omega)\sin\bar{\bar{\omega}} \tag{133}$$

and

$$\bar{e}\sin\bar{\omega} = (\bar{\bar{e}} + \Delta e)\sin\bar{\bar{\omega}} + (\bar{e}\,\Delta\omega)\cos\bar{\bar{\omega}}\ . \tag{134}$$

(Recovery of $\bar{e}$ from $\bar{e}\cos\bar{\omega}$ and $\bar{e}\sin\bar{\omega}$ is trivial; recovery of $\bar{\omega}$ involves, in Fortran, the use of the ATAN2 function, unless $\bar{e}$ is zero, in which case the arbitrary value of zero may be adopted for $\bar{\omega}$.)

The value of $\bar{M}$ still has to be found, but this is easy, if the non-singular ΔU is known, since it is only necessary to match $\bar{M}$ to ΔU . Thus the formula in terms of ΔU is

$$\bar{M} = (\bar{\bar{M}} + \bar{\bar{\omega}}) + \Delta U - \bar{\omega}\ , \tag{135}$$

in which any indeterminacy in $\bar{\omega}$, as derived from (133) and (134), is properly reflected in the preservation of a fully accurate $\bar{U}$.

The fact that $\bar{e}$ and $\bar{\omega}$ are legitimately derived in such a simple manner will be considered further in Part 3. It is assumed, of course, that Δe *etc* are evaluated, following sub-section (a), by integration of $\dot{e}_{\ell p}$ (*etc*); $\dot{e}_{\ell p}$ is given, via equation (72), by $e_{2,\ell p}$ and $e_{3,\ell p}$ (assuming only J_2 and J_3 to be operative), and these are obtained in sections 5 and 7. The essential point about the first-order Taylor expansions (133) and (134) is that they effectively allow for first-order $\Delta\xi$ and $\Delta\eta$ faithfully, without contamination by second-degree combinations of Δe and $\Delta\omega$ that would introduce intolerable error when $\Delta\omega$ is unbounded; a geometrical way of looking at this is that the formulae 'straighten out' the effect of an $\bar{e}\,\Delta\omega$ which, near the origin of the $(\bar{\xi}, \bar{\eta})$-plane, has a pronounced curvature.

It was indicated in section 2.1 that, for analysis in which there is no e-truncation, U is not a natural parameter to use, and this is because (as we shall see) the direct addition of ΔM and $\Delta\omega$ does not lead to simple expressions. Simple (and hence natural) expressions are given by $\Delta M + q\,\Delta\omega$, or $\Delta U'$ say, and on this basis (135) should be replaced by

$$\bar{M} = (\bar{\bar{M}} + \bar{q}\,\bar{\bar{\omega}}) + \Delta U' - \bar{q}\,\bar{\omega} , \qquad (136)$$

but an awkward point now arises, *viz* that when $\Delta\omega$ is large (because $\bar{e}$ is small) an ambiguity of 2π in the recovery of $\bar{\omega}$ from (133) and (134) can lead to an error of $2\pi(1 - \bar{q})$ in the derivation of $\bar{M}$ from (136). There is a similar difficulty in relation to the inclination singularity, and a double difficulty when the singularities occur together, as we shall see in sub-sections (c) and (d). It is enough to remark, in the present Report, that the procedure proposed in Ref 4 for resolving these ambiguities was inadequate, and new resolution criteria will be given in Part 3.

(c) Singularity avoidance for near-equatorial orbits

The situation for near-equatorial orbits closely parallels that for near-circular orbits, $\Omega_{3,\ell p}$ and $\omega_{3,\ell p}$ being infinite when s is zero. It is now the concept of 'node' that ceases to be meaningful, and with direct orbits the difficulty would vanish if we replaced the elements i, Ω and ω by (paralleling ξ, η and U) the quantities ξ, η and $\tilde{\omega}$, equal to $\sin i \sin \Omega$, $-\sin i \cos \Omega$ and $\omega + \Omega$, respectively. The use of $\tilde{\omega}$ has been traditional in celestial mechanics, where orbits are almost invariably direct, but it is inappropriate for retrograde orbits, for which it would be necessary to redefine $\tilde{\omega}$ as $\omega - \Omega$; furthermore, as with the addition of ΔM and $\Delta\omega$ to form ΔU , the perturbations in ω and Ω do not naturally either add or subtract directly. Both difficulties (different régimes and unnaturalness) disappear if we use the generalized quantity ψ, defined at the differential level by (31), as the replacement for ω .

As with ξ, η and U' , it is only necessary to use ξ, η and ψ implicitly in the relations corresponding to (133), (134) and (136), *viz*

$$\bar{s} \sin \Omega = \sin (\bar{\bar{i}} + \Delta i) \sin \bar{\bar{\Omega}} + (\bar{s}\,\Delta\Omega) \cos \bar{\bar{\Omega}} , \qquad (137)$$

$$- \bar{s} \cos \bar{\Omega} = - \sin (\bar{\bar{i}} + \Delta i) \cos \bar{\bar{\Omega}} + (\bar{s}\,\Delta\Omega) \sin \bar{\bar{\Omega}} \qquad (138)$$

and

$$\bar{\omega} = (\bar{\bar{\omega}} + \bar{c}\bar{\bar{\Omega}}) + \Delta\psi - \bar{c}\bar{\Omega} . \qquad (139)$$

As with $\bar{e}\,\Delta\omega$, $\bar{s}\,\Delta\Omega$ is to be regarded as a single entity in (137) and (138). Clearly, $\bar{s}$ and $\bar{\Omega}$ can be recovered at once from these equations, but a difficulty arises in the recovery of $\bar{i}$ from $\bar{s}$, since near-polar orbits have to be covered as well as near-equatorial ones. The inaccuracy in then deriving $\bar{i}$ from $\bar{s}$ has already been referred to (in section 2.1) and an extreme situation could arise in which the computed value of $\bar{s}$ became greater than unity. The problem is easily dealt with, however, at the expense of a minuscule amount of extra computing, by making use of $\bar{c}$, given by $\cos(\bar{\bar{i}} + \Delta i)$, as well as $\bar{s}$; this amounts to operating with all three components of the unit vector (ξ, η, ζ) . (See also Ref 7.)

It is worth remarking that the quantities ξ and η turn out to be 'less natural' than the corresponding ξ and η , associated with e and ω , in that the perturbation expressions lack the homogeneity (in regard to occurrences of the factor c) that might have been expected. This can be seen, for example, when (149) and (151) are combined, to give ξ_1 and η_1 and (as will be remarked again in section 4) the inhomogeneity is apparent. Two pairs of quantities exist that *are* homogeneous, however, *viz* $\tan \frac{1}{2}i \cos \Omega$ and $\tan \frac{1}{2}i \sin \Omega$, the pair appropriate for direct orbits, and $\cot \frac{1}{2}i \cos \Omega$ and $\cot \frac{1}{2}i \sin \Omega$, the pair appropriate for retrograde orbits. These quantities belong to the two sets of 'equinoctial elements' introduced by Broucke and Cefola[21], one set being fully non-singular for direct orbits and the other set fully non-singular for retrograde orbits. (The remaining equinoctial elements are a, $e \cos \tilde{\omega}$, $e \sin \tilde{\omega}$ and $M + \tilde{\omega}$, $\tilde{\omega}$ being, as defined earlier, $\omega \pm \Omega$ as appropriate.) An obvious advantage in redefining ξ and η to be a pair of equinoctial elements would be that no ζ would then be required, but the need for two different pairs is not very satisfactory. Further, there is really no disadvantage in the 'unnaturalness' of our ξ and η , since for explicit formulae (associated with J_2 and J_3) we are sticking to Δi and $\bar{s}\,\Delta\Omega$, ξ and η only being (implicitly) introduced in the general transformations (137) and (138). We *do* require explicit formulae for $\Delta\psi$, however, to avoid singular expressions, so it is only here that homogeneity is desirable; it seems ironic, therefore, that equinoctial elements are 'natural' where it does not matter, but 'unnatural' ($\tilde{\omega}$ and $M + \tilde{\omega}$) where it does matter.

There is a more basic respect in which the parallel between the eccentricity and inclination singularities breaks down, since the variation of ω constitutes the very essence of the $\Delta\zeta$, by definition, the variation of Ω being irrelevant. Thus the study of $\bar{e}$ and $\bar{\omega}$ (or $\bar{\xi}$ and $\bar{\eta}$) is essentially self-contained, whereas the study of $\bar{i}$ and $\bar{\Omega}$ cannot be divorced from

$\tilde{\omega}$. (In another sense, of course, the reverse of this is true, as i and Ω define the orbital plane, and its node, independently of ω , whereas ω only defines the perigee relative to the node.) The importance of the distinction should be clearer when these studies are pursued in Part 3.

(d) Avoidance of all singularity

Putting together the results of the previous sections, we can now summarize an approach that in principle avoids all singularity. The conceptual U' , of sub-section (b), must be replaced by L , where $\dot{L}$ is given by (35), and our first step is the evaluation of the five specific long-period perturbations Δe , Δi , $\bar{s}\,\Delta\Omega$, $\tilde{e}\,\Delta\psi$ and ΔL , using the expressions (of type $\xi_{2,\ell p}$ and $\xi_{3,\ell p}$) to be derived in sections 6 and 7; the evaluation is based on (129) and (130), with (131) and (132) also invoked for the induced contributions to $\Delta\Omega$ and $\Delta\omega$; since these contributions are always non-singular (free of $\bar{s}$ and $\tilde{e}$ divisors) they can most conveniently be incorporated directly with the purely secular generation of $\bar{\bar{\Omega}}$ and $\bar{\bar{\omega}}$ (required together with that of $\bar{\bar{M}}$).

The values of $\tilde{i}$ and $\tilde{\Omega}$ can now be obtained, in principle from (137) and (138) but with $\tilde{c}$ used as well as $\bar{s}$, as has been described. We next, again in principle, derive $\tilde{e}$ and $\tilde{\psi}$ from versions of (133) and (134) in which ω has been replaced by ψ , with $\bar{\bar{\psi}}$ set to $\bar{\bar{\omega}} + \bar{c}\bar{\bar{\Omega}}$; then $\tilde{\omega}$ is obtained from (139), re-expressed simply as $\tilde{\omega} = \tilde{\psi} - \bar{c}\tilde{\Omega}$. The arbitrary nature of ψ imposes no difficulty in the use of $\cos\bar{\bar{\psi}}$ and $\sin\bar{\bar{\psi}}$, since (133) and (134) are preserved under the addition of an arbitrary quantity to $\bar{\bar{\omega}}$, and in fact it is convenient to use versions of these formulae in which $\bar{\bar{\omega}}$ is replaced by $\bar{\bar{\omega}} + \tilde{c}\,(\bar{\bar{\Omega}} - \tilde{\Omega})$, instead of $\bar{\bar{\psi}}$, since $\tilde{\omega}$ is then obtained directly; however, this does not get round the difficulty associated with the 2π ambiguity in $\bar{\bar{\Omega}} - \tilde{\Omega}$, and we return to this in Part 3.

It remains to derive $\tilde{M}$. The formula for this is a modified form of (136), *viz*

$$\tilde{M} = (\bar{\bar{M}} + \bar{q}\bar{\bar{\psi}}) + \Delta L - \bar{q}\tilde{\psi} , \qquad (140)$$

and again the only difficulty is the one associated with the 2π ambiguity in $\bar{\bar{\psi}} - \tilde{\psi}$.

4 FIRST-ORDER ANALYSIS FOR J_2 PERTURBATIONS

4.1 Perturbations in the osculating elements

Substitution of the potential given by (48) into the planatary equations, (22)-(27), gives exact equations for the variation of the osculating elements (excluding M); expressed with the help of the C_j and S_j families given by (19), these equations are

$$\dot{a} = -\frac{Kna}{2q^5}\left(\frac{p}{r}\right)^4 \left\{f(5eS_3 + 4S_2 - eS_1) + 4eh \sin v\right\}, \tag{141}$$

$$\dot{e} = -\frac{Kn}{8q^3}\left(\frac{p}{r}\right)^3 \left\{f(5eS_4 + 14S_3 + 12eS_2 + 2S_1 - eS_0) + 4h(e \sin 2v + 2 \sin v)\right\}, \tag{142}$$

$$\dot{i} = -\frac{Kn}{q^3}\left(\frac{p}{r}\right)^3 s\,c\,S_2 , \tag{143}$$

$$\dot{\Omega} = \frac{Kn}{q^3}\left(\frac{p}{r}\right)^3 c(C_2 - 1) , \tag{144}$$

$$\dot{\omega} = \frac{Kn}{8eq^3}\left(\frac{p}{r}\right)^3 \left\{f(5eC_4 + 14C_3) - 2e(4 - 7f)\,C_2 - f(2C_1 - eC_0) + 4h(e \cos 2v + 2 \cos v) + 2e(6 - 7f)\right\} \tag{145}$$

and

$$\dot{\sigma} = -\frac{Kn}{8eq^2}\left(\frac{p}{r}\right)^3 \left\{f(5eC_4 + 14C_3 - 18eC_2 - 2C_1 + eC_0) + 4h(e \cos 2v + 2 \cos v - 3e)\right\} . \tag{146}$$

The independent variable is now changed from t to $\tilde{v}$ (or strictly $\tilde{v}$), via (39), after which the first-order integration of each equation is straightforward. It is noted that v (in one form or another) acts as the only variable in each integrand, being involved in three different ways: (i) explicitly;

(ii) via the C_j and S_j (for $j \neq 0$) by definition; and (iii) via $\bar{p}/\bar{r}$ and equation (63). For ease of expression, we will from now drop bars (and tildes) in $\bar{v}$, etc, wherever this should cause no confusion. (To first order, the results of this section are correct without the bars, but bars must be added to all first-order expressions when we extend to second order.)

The results of the integration are as follows, when expressed in terms of the notation introduced by (71) and (73), by using the bracket conventions of section 2.1, and after adoption of the best set of integration constants (as discussed in section 3.1 and to be justified in section 4.3):

$$a_1 = \frac{1}{24} a q^{-2} \Big\{ 3f\,(e^3 C_5 + 6e^2 C_4 + 3e[4,1]C_3 + 4[2,3]C_2 + 3e[4,1]C_1 + 6e^2 C_0 + e^3 C_{-1}) + 4h\,(e^3 \cos 3v + 6e^2 \cos 2v + 3e[4,1] \cos v + 2[2,3]) \Big\}, \tag{147}$$

$$e_1 = \frac{1}{48} \Big\{ f\,(3e^2 C_5 + 18eC_4 + [28,17]C_3 + 60eC_2 + 3[4,11]C_1 + 18eC_0 + 3e^2 C_{-1}) + 4h\,(e^2 \cos 3v + 6e \cos 2v + 3[4,1] \cos v + 10e) \Big\}, \tag{148}$$

$$i_1 = \frac{1}{6} sc\,(eC_3 + 3C_2 + 3eC_1 + 3), \tag{149}$$

$$\hat{\Omega}_1 = -c, \tag{150}$$

$$\Omega_1 = \frac{1}{6} c\,(eS_3 + 3S_2 + 3eS_1 - 6e \sin v), \tag{151}$$

$$\hat{\omega}_1 = 2g, \tag{152}$$

$$\omega_1 = \frac{1}{48} e^{-1} \Big\{ 3e^2 fS_5 + 18efS_4 + [28f, -(8,-19)]S_3 - 12e(2,-5)S_2 - 3[4f,(8,-15)]S_1 - 18efS_0 - 3e^2 fS_{-1} + 4e^2 h \sin 3v + 24eh \sin 2v + 6[8h,(14,-17)] \sin v \Big\}, \tag{153}$$

$$\hat{\sigma}_1 = qh \tag{154}$$

and

$$\sigma_1 = -\frac{1}{48} e^{-1} q \left\{ f \left(3e^2 S_5 + 18eS_4 + [28,-13]\, S_3 - 36eS_2 - 3[4,17]\, S_1 - 18eS_0 - 3e^2 S_{-1}\right) + 4h(e^2 \sin 3v + 6e \sin 2v + 3[4,-5] \sin v) \right\} . \tag{155}$$

A certain inconformity will be observed in (153), in comparison with (148) and (155) in particular, in that we do not have the same neat division into 'pure f-terms', involving the C_j, and 'pure h-terms', involving $\cos jv$. This inconformity, propagated from (145), follows from the presence of the term in $\partial U/\partial i$ in (26), and the natural partner of (148) is not (153) but (166), *ie* the expression for ψ_1 given in the next section, since (33), the equation for $\dot{\psi}$, does not contain $\partial U/\partial i$. (For further comments on this, see Part 2.)

It only remains to obtain the perturbation in M (effectively to supersede the perturbation in σ), the basis for the analysis having been given in section 2.2. As explained in section 3.1, this in principle involves the derivation of a modified version of Kepler's third law, but it turns out that no modification is needed in practice; this is fortuitous, and it does not apply for any other J_ℓ. Thus we require nothing but the formula for M_1 itself. The starting point is (40), and to use this we require a suitably expressed formula for n_1, the derivation of which calls for some preliminary discussion of (147), the formula for a_1.

As explained in sections 2.3 and 3.1, the integration constant in (147), part of which appears in the 3f-component of the equation and part in the 4h-component, is such as to make the mean element $\bar{a}$ identical with the exact constant a' given by the conservation of energy. But a' is related to (osculating) a by the exact equation (52) which, on substitution for the potential given by (43), gives

$$a_1 = \frac{1}{3} aq^{-2} \left(\frac{p}{r}\right)^3 (3fC_2 + 2h) . \tag{156}$$

This is in non-standard form, because of the factor $(p/r)^3$, but expansion of this, by use of (5), at once confirms (147) and incidentally explains the manifest symmetry there of the C_j terms - the 3f-component of the integration cosntant, *viz* $6e^2C_0$, is clearly essential for this symmetry, to match the term $6e^2C_4$. Of the two (non-standard) expressions, intermediate in powers of p/r , that emerge as (156) expands to (147), the first is the one corresponding to the required formula for n_1 , but the second has an application as well, to be met in section 5.9. The first expression is

$$a_1 = \frac{1}{6} aq^{-2} \left(\frac{p}{r}\right)^2 \left\{ 3f\,(eC_3 + 2C_2 + eC_1) + 4h(1 + e \cos v) \right\} \tag{157}$$

and the second is

$$a_1 = \frac{1}{12} aq^{-2} \left(\frac{p}{r}\right) \left\{ 3f\,(e^2C_4 + 4eC_3 + 2[2,1]C_2 + 4eC_1 + e^2C_0) + 4h\,(e^2 \cos 2v + 4e \cos v + [2,1]) \right\} . \tag{158}$$

To each of the four expressions for a_1 there is an immediately corresponding expression for n_1 , given by

$$n_1 = -\left(\frac{3n}{2a}\right) a_1 , \tag{159}$$

on the provisional assumption that the $\bar{n}$ we require is indeed such that the Kepler law is satisfied without modification. The reason why (157) leads to the most apposite expression for n_1 is that the factor $(p/r)^2$ then cancels out when δn (derived from this expression) is substituted in (40), W in (40) being defined by (17).

Thus we have

$$n_1/nW = -\frac{1}{4} q \left\{ 3f\,(eC_3 + 2C_2 + eC_1) + 4h\,(1 + e \cos v) \right\} , \tag{160}$$

which, in view of (40), is in the right form for further integration. The integration leads to a secular term, as well as short-period terms, and the

resulting expressions, with the usual notation involving the symbol introduced in (3), are given by

$$\hat{\int}_1 = -qh \tag{161}$$

and

$$\int_1 = -\frac{1}{4}q\left\{f(eS_3 + 3S_2 + 3eS_1) + 4h\,e\sin v\right\}. \tag{162}$$

The secular perturbation in M is now given by the combination of (154) with (161), and the short-period perturbation by the combination of (155) with (162). The former combination gives

$$\hat{M}_1 = 0 \tag{163}$$

as anticipated, validating the provisional assumption about Kepler's third law for mean elements, whilst the latter combination gives

$$M_1 = -\frac{1}{48}e^{-1}q\left\{f(3e^2S_5 + 18eS_4 + [28,-1]S_3 - 3[4,5]S_1 - 18eS_0 - 3e^2S_{-1}) + 4h(e^2\sin 3v + 6e\sin 2v + 3[4,-1]\sin v)\right\}. \tag{164}$$

4.2 Perturbations in related quantities

For reference, we give formulae for perturbations in a number of quantities related to the standard osculating elements covered in the preceding section; we start with the conceptual ψ, ρ and L, whose rates of change are given by (33), (34) and (35). We have

$$\hat{\psi}_1 = h, \tag{165}$$

$$\psi_1 = \frac{1}{48}e^{-1}\left\{f(3e^2S_5 + 18eS_4 + [28,11]S_3 + 36eS_2 - 3[4,-7]S_1 - 18eS_0 - 3e^2S_{-1}) + 4h(e^2\sin 3v + 6e\sin 2v + 3[4,3]\sin v)\right\}, \tag{166}$$

$$\hat{\rho}_1 = 2hq , \tag{167}$$

$$\rho_1 = \frac{1}{2} q \left\{ f(eS_3 + 3S_2 + 3eS_1) + 4he \sin v \right\} , \tag{168}$$

$$\hat{L}_1 = hq \tag{169}$$

and

$$L_1 = \frac{1}{4} q \left\{ f(eS_3 + 3S_2 + 3eS_1) + 4he \sin v \right\} . \tag{170}$$

In practice, of course, the ψ and ρ results were obtained *before* the ω and σ results, (152)-(155) being derived from (150)-(151) and (165)-(168). Further, it will be noted that $\rho_1 = -2\int_1$ and $L_1 = -\int_1$ (and likewise $\hat{\rho}_1 = -2\hat{\int}_1$ and $\hat{L}_1 = -\hat{\int}_1$); these are not coincidental results, and it may be seen from Refs 1 and 22 (see also Part 2) that there is a general first-order result, for the zonal harmonic J_ℓ, given by $\delta\rho = -\frac{2}{3}(\ell + 1)\,\delta\!\int$ and $\delta L = -\frac{1}{3}(2\ell - 1)\,\delta\!\int$.

Next, we can use (147) and (148) to derive p_1 , since, from (28)

$$p_1 = q^2 a_1 - 2ae\, e_1 . \tag{171}$$

Thus

$$p_1 = \frac{1}{3} p \left\{ f(eC_3 + 3C_2 + 3eC_1) + 2h \right\} . \tag{172}$$

We can employ the last two equations in reverse, to obtain a more compact (but non-standard) expression for e_1 based on the most compact of the four formulae for a_1 , *viz* (156). This expression, which can be confirmed by direct reduction from (148) and which closely corresponds to the first-order expression for δe given by Kozai[17], is

$$e_1 = \frac{1}{6} e^{-1} \left\{ \left(\frac{p}{r}\right)^3 (3fC_2 + 2h) - q^2 \left[f(eC_3 + 3C_2 + 3eC_1) + 2h \right] \right\} . \tag{173}$$

(Though compact, the expression is unfortunate in having the factor e^{-1} ; as can easily be seen, however, this cancels out when the bracketed terms are expanded.)

Again, since

$$(pc^2)_1 = c^2 p_1 - 2psc\, i_1 \ , \tag{174}$$

(149) and (172) lead to

$$(pc^2)_1 = \tfrac{2}{3} pc^2 (1 - 3f) \ . \tag{175}$$

This is a (first-order) constant, as it must be, since (as was remarked in section 2.2) pc^2 is an exact constant for a geopotential that involves only the zonal harmonics; it represents the polar component of the satellite's angular momentum. The reason that $(pc^2)_1$ does not actually vanish is that the constants of integration for a, e and i are not (optimally) such as to force this to happen.

Perturbations in ξ and η, defined by (9) and (10), can be obtained using (148), (152) and (153), but it would actually be unhelpful to use (152) to introduce quantities $\hat{\xi}$ and $\hat{\eta}$ because this would involve the conversion of a straightforward (purely secular) $\hat{\omega}$ into an apparently long-periodic variation (cf remarks in section 3.5(b)). However, the pure short-period perturbations can usefully be expressed with the help of the γ_j, σ_j, c_j and s_j families, given by (18) and (20) (and so far unused); in terms of these, (148) and (153) lead to

$$\xi_1 = \tfrac{1}{48} \{ 3e^2 f\gamma_5 + 18ef\gamma_4 + 2\,[14f, -(2,-9)]\,\gamma_3 - 12e(1,-5)\gamma_2 - 3e^2(4,-13)\gamma_1$$
$$+ e^2(9,-7)c_3 + 36e(1,-1)c_2 + 6\,[2(4,-5), (10,-11)]\,c_1$$
$$+ 2e(20,-21)c_0 - 9e^2(4,-5)\,c_{-1} \}$$

......(176)

and

$$\eta_1 = \tfrac{1}{48} \{ 3e^2 f\sigma_5 + 18ef\sigma_4 + 2\,[14f, -(2,-9)]\,\sigma_3 - 12e(1,-5)\sigma_2 - 3e^2(4,-13)\sigma_1 - 5e^2 fs_3$$
$$+ 12e(1,-3)s_2 + 6[2(4,-7), 3(2,-3)]s_1 + 2e(20,-39)s_0 - 3e^2(12,-13)s_{-1} \} .$$

......(177)

Formulae for ξ_1 and η_1 are not given, because (effectively due to the c-factor in i_1) i_1 and Ω_1 do not combine in a natural (homogeneous) way. This was remarked in section 3.5(c), where it was also remarked that, for the equinoctial elements, i_1 and Ω_1 *do* combine homogeneously. The resulting formulae, for both pairs of equinoctial elements, are:

$$(\tan \tfrac{1}{2}i \cos \Omega)_1 = \frac{\sin i \cos i}{6(1 + \cos i)} \{3 \cos \Omega - 3e \cos (v + \Omega) + 3e \cos (v - \Omega) + e \cos (3v + \Omega + 2\omega) + 3 \cos (2v + \Omega + 2\omega) + 3e \cos (v + \Omega + 2\omega)\} , \tag{178}$$

$$(\tan \tfrac{1}{2}i \sin \Omega)_1 = \frac{\sin i \cos i}{6(1 + \cos i)} \{3 \sin \Omega - 3e \sin (v + \Omega) - 3e \sin (v - \Omega) + e \sin (3v + \Omega + 2\omega) + 3 \sin (2v + \Omega + 2\omega) + 3e \sin (v + \Omega + 2\omega)\} , \tag{179}$$

$$(\cot \tfrac{1}{2}i \cos \Omega)_1 = -\frac{\sin i \cos i}{6(1 - \cos i)} \{3 \cos \Omega - 3e \cos (v - \Omega) + 3e \cos (v + \Omega) + e \cos (3v - \Omega + 2\omega) + 3 \cos (2v - \Omega + 2\omega) + 3e \cos (v - \Omega + 2\omega)\} \tag{180}$$

and

$$(\cot \tfrac{1}{2}i \sin \Omega)_1 = -\frac{\sin i \cos i}{6(1 - \cos i)} \{3 \sin \Omega + 3e \sin (v - \Omega) + 3e \sin (v + \Omega) - e \sin (3v - \Omega + 2\omega) - 3 \sin (2v - \Omega + 2\omega) - 3e \sin (v - \Omega + 2\omega)\} . \tag{181}$$

For the remaining quantities covered in this section, it would be even less helpful to consider long-term perturbations than for ξ and η , but pure short-period perturbations can again be usefully expressed. Thus, v_1 can be derived from (148) and (164), using (42); the result is

$$v_1 = -\frac{1}{48} e^{-1}\Big\{f\big(3e^2S_5 + 18eS_4 + [28,11]S_3 + 32eS_2 - [12,-5]S_1 - 18eS_0 - 3e^2S_{-1}\big) + 4h\big(e^2 \sin 3v + 6e \sin 2v + [12,1] \sin v\big)\Big\} . \quad (182)$$

From v_1 and ω_1 , the latter given by (153), the important expression for u_1 can be obtained; it is

$$u_1 = -\frac{1}{12}\Big\{2e(1,-1)S_3 + (6,-7)S_2 + 2e(3,-5)S_1 - 4e(5,-6) \sin v\Big\} . \quad (183)$$

The expressions for ξ_1, η_1 and u_1 can be used for a good overall check that also invoves the expression for r_1 in the next section. Thus, from the formula

$$e \cos v = \xi \cos u + \eta \sin u \quad (184)$$

it follows that

$$(e \cos v)_1 = \xi_1 \cos u + \eta_1 \sin u - u_1(\xi \sin u - \eta \cos u) , \quad (185)$$

which evaluates to

$$(e \cos v)_1 = \frac{1}{24}\Big\{f\big(3e^2C_4 + 16eC_3 + 2[10,7]C_2 + 32eC_1 + 11e^2C_0\big) + 4h\big(e^2 \cos 2v + 8e \cos v + [6,1]\big)\Big\} . \quad (186)$$

The factor $(1 + e \cos v)$ can be extracted from the right-hand side of (186) and the resulting expression leads, in view of (5), to

$$(p/r)_1 = \frac{1}{12} (p/r)\Big\{f\big(3eC_3 + 10C_2 + 11eC_1\big) + 4h(e \cos v + 3)\Big\} . \quad (187)$$

This result ties up immediately with p_1 , given by (172), and r_1 given by (188).

A certain property may be observed in the structure of the various expressions given in sections 4.1 and 4.2. For a fully non-singular quantity,

ζ, the property for ζ_1 is that the coefficient of the sine or cosine of $jv + ku$ ($= j'v + k\omega$, with $j' = j + k$) contains (at least) the factors $e^{|j|}$ and $s^{|k|}$; if it were not so, a change to non-singular variables (involving $e \cos v$, $e \sin v$, $s \cos u$ and $s \sin u$) would result in uneliminated singular terms, via a negative power of e or s. The property operates in the first-order perturbation analysis for an arbitrary zonal harmonic, and it is related to the so-called d'Alembert characteristic[12]. It may be seen to apply here with a_1, ρ_1, L_1 and p_1.

The property does not apply directly to e_1 and i_1, because the elements e and i, though non-singular in themselves, do not have well-behaved rates of change at their associated singularities. (This is easily seen by considering the relation $e^2 = \xi^2 + \eta^2$, in particular.) Thus it is ee_1 and si_1, rather than e_1 and i_1, that have the property. For quantities that are *intrinsically* singular, the effect on ζ_- is naturally more severe, the property being satisfied for $s^2\Omega$, $s^2e^2\omega_1$, $e^2\sigma_1$, e^2M_1, $e^2\psi_1$, $s^3\xi_1$, $s^3\eta_1$, e^2v_1 and s^2u_1.

In the next section it will be found that r_1, b_1 and w_1 also have the property, this being an essential aspect of their merit of course. Finally, for the second-order formulae to be developed in sections 5 and 6, where modification of the direct property relates to *second*-order rates of change, the property will be found to hold, in particular, for a_2, e^3e_2, s^3i_2, p_2, $s^4\Omega_2$, $s^4e^4\omega_2$, e^4M_2, r_2, b_2 and w_2.

4.3 Perturbations in spherical coordinates

As explained in section 3.3, the complicated expressions for short-period perturbations in the osculating elements (as given in section 4.1) can be combined into compact and convenient expressions for perturbations in the system of spherical coordinates (r, b, w). Expressions for *pure* short-period perturbations can be obtained from the six basic ζ_1 of section 4.1 by substitution in (95)-(97). The results are worth writing with unabbreviated sines and cosines, in view of their importance, and in terms of explicit mean and semi-mean elements; thus

$$r_1 = \frac{1}{6}\bar{p}(\bar{f}\cos 2\tilde{u} - 2\bar{h}), \tag{188}$$

$$\rho_1 = \frac{1}{3}\bar{e}\bar{s}\bar{c}\left\{\sin(\tilde{u} + \tilde{v}) - 3\sin\tilde{\omega}\right\} \tag{189}$$

and

$$w_1 = \frac{1}{12}\{\bar{f}[\sin 2\tilde{u} + 4\bar{e}\sin(\tilde{u}+\tilde{\omega})] + 8\bar{e}\bar{h}\sin\tilde{v}\} . \tag{190}$$

The independent check of (188), based on (187), has been mentioned. There is no obvious check of (189) or (190), however - verification that (94) is satisfied proves little, since (94) was used directly in the derivation of (97).

We can now justify the claim that we are using the best set of integration constants* for the ζ_1 . With any other choice of constant for a_1 , (188) would (via (95)) contain a complicating term in r/a . With any other constants for e_1 and M_1 , (188) would also contain a term combining $\cos v$ and $\sin v$, and there would be considerable complication in (190). With any other constants for i_1 and Ω_1 , (189) would, likewise, contain a term combining $\cos u$ and $\sin u$. Finally, any other constant in ω would be transmitted directly into (190).

On differentiating (188)-(190), we have expressions for $\dot{r}_1$, $\dot{b}_1$ and $\dot{w}_1$ immediately. Thus

$$\dot{r}_1 = -\frac{1}{3}\bar{p}\bar{f}\dot{\tilde{u}}\sin 2\tilde{u} , \tag{191}$$

$$\dot{b}_1 = \frac{1}{3}\bar{e}\bar{s}\bar{c}\{(\dot{\tilde{u}}+\dot{\tilde{v}})\cos(\tilde{u}+\tilde{v}) - 3\dot{\tilde{\omega}}\cos\tilde{\omega}\} \tag{192}$$

and

$$\dot{w}_1 = \frac{1}{6}\{\bar{f}[\dot{\tilde{u}}\cos 2\tilde{u} + 2\bar{e}(\dot{\tilde{u}}+\dot{\tilde{\omega}})\cos(\tilde{u}+\tilde{\omega})] + 4\bar{e}\bar{h}\dot{\tilde{v}}\cos\tilde{v}\} . \tag{193}$$

For a purely first-order solution we can set $\dot{\tilde{\omega}} = 0$ and $\dot{\tilde{v}} = \dot{\tilde{u}} = \bar{W}\bar{n}$ in these expressions. For incorporation in the second-order solution, however, we must allow for the first-order variation in $\tilde{\omega}$; we deal with $\dot{\tilde{u}}$ just by identifying it with $\dot{\tilde{w}}'$, as given by (113) - *cf* the remark after that equation.

---

* Some of the 'constants' are represented by two distinct terms in the ζ_1 ; in (147), *eg*, there is a term in C_0 , which contributes a first-order constant to Ka_1 , whilst the final term may be regarded as a 'pure constant', making a contribution that is constant to second order. In the $J_2{}^2$ analysis, similarly, ζ_2 may contain three distinct 'constant' terms, of which the first two are 'less constant' than the third.

Equations (188)-(190) lead at once to δr, δb and δw, since there are no $\zeta_{1,\ell p}$ in (74). If we want the complete short-period perturbations (δr, δb and δw), we must allow for the carry-over effects of $\hat{\Omega}_1$ and $\hat{\omega}_1$. The resulting formulae are given by

$$\delta r = K r_1 , \tag{194}$$

$$\delta b = K\left(b_1 + \bar{c}\,\bar{s}\,\bar{m} \cos \tilde{u}\right) , \tag{195}$$

$$\delta w = K\left(w_1 + \bar{h}\bar{m}\right) . \tag{196}$$

The full expressions can easily be shown to be equivalent to the first-order terms of (366)-(368) of Ref 3, so long as an error in (366) *ibid* is corrected by changing $\ldots\bar{K}\bar{a}\{(\bar{f}\bar{q}^2 \cos 2\tilde{u} \ldots$ into $\ldots\bar{K}\bar{p}\{(\bar{f} \cos 2\tilde{u} \ldots$. A second error in (366) *ibid* is worth remarking, *viz* that $\ldots + 4(4 - 12f - 19f^2)\ldots$ should read $\ldots + (16 - 48\bar{f} + 23\bar{f}^2)\ldots$; this pair of errors (in the same equation) are, apart from rather obvious ones in equations (35) and (81), the only errors known in Ref 3. (The quoted equations from Ref 3 are from the 'Note added in proof'; this was written after the author suddenly registered the superiority of a' as mean semi-major axis, having started by recommending an $\bar{a}$ only $O(Ke^2)$ different from Kozai's $\bar{a}$.) From (194)-(196) we can get the corresponding expressions for $\delta\dot{r}$, $\delta\dot{b}$ and $\delta\dot{w}$. For the reason indicated in section 3.2, however, the first-order solution with carry-over effects included is not a good basis for development of the complete second-order solution; so it is best to regard (194)-(196) as of only academic interest - the important equations in this section are (188)-(190) and (191)-(193).

5 $J_2{}^2$ PERTURBATIONS IN OSCULATING ELEMENTS

Before proceeding to the second-order solution for the J_2-only field, it is worth stressing again that the first-order solution given in section 4 should in general be understood to be expressed in terms of semi-mean quantities and not osculating ones. Thus the arguments of the trigonometric functions should be interpreted as if tildes were carried, and the coefficients of these functions as if bars were carried, there being no distinction, for a, e and i , between mean and semi-mean elements. The omission of bars and tildes was a matter of visual convenience, and was in any case legitimate when second-order terms were not under consideration.

We can summarize the first-order solution for the generic osculating element as follows. For Ω and ω only, there is a secular perturbation given by $\bar{K}\hat{\zeta}_1\bar{n}t$ (*cf*(71)), that propagates the mean element $\bar{\zeta}$, together with a carry-over effect, $\bar{K}\hat{\zeta}_1\bar{m}$, that (added to $\bar{\zeta}$) yields the semi-mean element $\tilde{\zeta}$. There are no long-period perturbations, but for each element there is a pure short-period perturbation given by $\bar{K}\zeta_1$ (*cf* (73)).

The second-order solution for ζ comprises the first-order terms just summarized and the second-order terms now to be developed. The understanding about the omitted bars and tildes in the first-order terms now becomes vital, but (because we are not concerned with third-order terms) it will be legitimate (and certainly easier on the eye) if bars and tildes are omitted in the new terms wherever possible. In addition to the second-order secular and short-period perturbations specified by $\hat{\zeta}_1$ and $\hat{\zeta}_2$ respectively, long-period perturbations will now be making their appearance. Once the expressions for the $\zeta_{2,\ell p}$ are available (*cf* (72)) - and they are summarized in section 6.1 - the $J_2{}^2$-component of each $\Delta\zeta$ is effectively known, since the method of generating it was given in section 3.5(a). Further, it will be found that these components of the $\Delta\zeta$ (unlike those associated with J_3) are all non-singular. The carry-over effects of the $\zeta_{2,\ell p}$ must not be forgotten; they are incorporated in the $\delta\zeta$ (cf (74)) and the amalgamation of the $\delta\zeta$ into δr, δb and δw is the subject matter of sections 6.2 to 6.4.

The basic idea in deriving expressions for the $\hat{\zeta}_2$, $\zeta_{2,\ell p}$ and ζ_2 is to 'bootstrap' on the first-order solution by substituting it in the right-hand sides of the planetary equations and then re-integrating. A special procedure, free of integration, is possible for $\zeta = a$, and this is developed in section 5.1, paralleling the derivation of (147) via (156). We then proceed, in section 5.2, to an effective rederivation of a_2 ($\hat{a}_2$ and $a_{2,\ell p}$ do not arise) by the general procedure, and extend it to the other elements in sections 5.3 to 5.10. It will be found, in sections 5.9 and 5.10, that $\bar{n}$ can no longer most conveniently be identified with n'.

5.1 Perturbation in a (special method)

We want a_2 such that

$$a = \bar{a} + \frac{1}{3}\bar{K}\bar{a}\bar{q}^{-2}(\bar{p}/\bar{r})^3(3\bar{f}\tilde{C}_2 + 2\tilde{h}) + \bar{K}^2 a_2 , \tag{197}$$

the starting point for which is the exact equation (52), which leads (still exactly) to (*cf* (156))

$$a = a' + \frac{1}{3} K a' q^{-2} (p/r)^3 (3 f C_2 + 2h) . \tag{198}$$

On comparing (197) and (198) it follows that there will be five sources of terms contributing to a_2 ; four of these sources are essentially the same as four of the five listed in Refs 2 and 3, but source (ii) of these no longer arises now that we have unreservedly identified $\bar{a}$ with a' , and it is replaced by a source that we could previously neglect (due to e-truncation). The sources are as follows: (i) the variation of p in K , which could easily be overlooked; (ii) the variation of e in q^{-2} (this is the new source); (iii) the variation of $(p/r)^3$; (iv) the variation of i , which affects f and hence h ; and (v) the variation of u in C_2 . The contributions of these sources are as follows, with bars and tildes suppressed in line with the general principle laid down:

(i) $$-\frac{2}{3} q^{-4} (p/r)^3 (3fC_2 + 2h) p_1 ;$$

(ii) $$\frac{2}{3} aeq^{-4} (p/r)^3 (3fC_2 + 2h) e_1 ;$$

(iii) $$aq^{-2} (p/r)^2 (3fC_2 + 2h) (p/r)_1 ;$$

(iv) $$2aq^{-2} (p/r)^3 sc(C_2 - 1) i_1 ;$$

(v) $$-2aq^{-2} (p/r)^3 fS_2 u_1 .$$

Here p_1, e_1 $(p/r)_1$, i_1 and u_1 are given by (172), (148), (187), (149) and (183) respectively. In using these quantities, we are effectively identifying $\delta\zeta$ with $\delta_p\zeta$ (as appropriate), this being legitimate as a_2 is to be multiplied by $\bar{K}^2$; the appearance of $\tilde{C}_2$, as opposed to $\bar{C}_2$, in (197) validates the use of u_1 (associated with δu and $\delta_p u$ rather than δu).

It is very convenient that (187) expresses $(p/r)_1$ in a form containing (p/r) as a factor, since this ensures that each source of a_2 contains $(p/r)^3$ as a factor. On multiplying out the remaining factors in each source, and combining the resulting expressions, we eventually derive the required result, viz that (with bars and tildes omitted)

$$
\begin{aligned}
a_2 = \tfrac{1}{144} a q^{-4} (1 + e \cos v)^3 \Big[3f^2 \big\{ & 3e^3\Gamma_7 + 18e^2\Gamma_6 + 15e[2,1]\Gamma_5 + 4[4,11]\Gamma_4 \\
& \cdot e[46,-1]\Gamma_3 + 18e^2\Gamma_2 + 3e^3\Gamma_1 \big\} \\
& + 12f \big\{ 2e^3hC_5 + 12e^2hC_4 + e[2(22,-31), -(14,-17)]C_3 \\
& \qquad + 8h[3,2]C_2 - e[2(2,3), -(34,-39)]C_1 \\
& \qquad\qquad + 12e^2hC_0 + 2e^3hC_{-1} \big\} \\
& + 2 \big\{ e^2(8h^2 + 9f^2)(e \cos 3v + 6 \cos 2v) \\
& \qquad + 3e[2(24,-56,49), -(8,8,-37)] \cos v \\
& \qquad\qquad + 2[2(20h^2 + 3f^2) + 39e^2f^2] \big\} \Big] ;
\end{aligned}
\qquad \ldots\ldots (199)
$$

the bracket conventions of section 2.1 are used in this result.

5.2 Perturbation in a (general method)

This section serves as a prototype for the application of the general 'bootstrapping' procedure to one orbital element after another. The fact that it permits a rederivation of (199) acts as a check on the correctness of the general method, and also on the algebraic reductions involved (for a) in both methods.

The starting point is the exact equation (141), converted into an equation for $da/d\bar{v}$ by use of (39). To have a valid basis for second-order analysis it is essential that we distinguish carefully between barred and unbarred quantities in this conversion, the required form of the equation being

$$\frac{da}{d\bar{v}} = -\tfrac{1}{2}Ka(n/\bar{n})(\bar{q}^3/q^5)\{(p/r)^4/(\bar{p}/\bar{r})^2\}\, Q_a , \qquad (200)$$

where

$$Q_a = f(5eS_3 + 4S_2 - eS_1) + 4eh \sin v . \qquad (201)$$

To express $da/d\bar{v}$ as a function of mean quantities only, we must eliminate the osculating quantities a (and the basically equivalent n), i, e, v and ω , and we do this by substituting the first-order solutions for these five quantities into the right-hand side of (200), via a first-order Taylor expansion relative to the corresponding (semi-)mean quantities. A special situation arises

in regard to ω (implicit in Q_a via S_3, S_2 and S_1, and *explicit* when each S_j is replaced by $\sin(jv + 2\omega)$), and this requires preliminary consideration.

First-order analysis effectively involved the identification of barred and unbarred quantities in (200) and (if we ignore for the moment the complication caused by the resulting factor $(p/r)^2$) the interpretation of each S_j in (201) as (zero-order) $\sin(j\bar{v} + 2\bar{\omega})$, with $\bar{\omega}$ taken as constant in the integration with respect to $\bar{v}$; the result of the integration was then naturally taken as $-(1/j)\cos(j\bar{v} + 2\bar{\omega})$, with $\bar{\omega}$ evaluated (*after* the integration) to allow for $\hat{\omega}_1$. In the second-order analysis we require a replacement for S_j that is first-order correct, *ie* given by $\sin\{j(\bar{v} + \bar{K}v_1) + 2(\bar{\omega} + \bar{K}\omega_1)\}$, but in the first-order component of the extended solution we will still not be making proper allowance for the variation of $\bar{\omega}$, since we want to go on taking the integral in this component to be $-\tilde{C}_j/j$; this means that there is a source of second-order perturbation (for each S_j in Q_a) additional to the direct Taylor effects associated with v_1 and ω_1 (*viz* given by $j\bar{K}\,\tilde{C}_j\,v_1$ and $2\bar{K}\,\tilde{C}_j\,\omega_1$ respectively). It is this additional source of perturbation that constitutes the 'special situation' we are considering, being associated with the error in taking the integral of $\tilde{S}_j$ to be $-\tilde{C}_j/j$ when (to second order) it is really $-\tilde{C}_j/(j + 2\bar{K}\hat{\omega}_1)$. But

$$\frac{1}{j + 2\bar{K}\hat{\omega}_1} = \frac{1}{j} - \frac{2\bar{K}\hat{\omega}_1}{j^2} + O(\bar{K}^2)\,, \tag{202}$$

so the error can be rectified by introducing the 'carry-over' term $(2\bar{K}\hat{\omega}_1/j^2)\,\tilde{C}_j$. Since this term is the integral (with respect of $\bar{v}$) of $(-2\bar{K}\hat{\omega}_1/j)\,\tilde{S}_j$ (neglecting terms that would carry over to third order!), we can cope with the 'special situation' by supposing that there is an '$\hat{\omega}$ contribution' to $da/d\bar{v}$, as well as the obvious ω contribution. This $\hat{\omega}$ contribution is such that each S_j in Q_a would (apart from the complication of the $(p/r)^2$ factor that has been ignored so far) make a contribution to $da/d\bar{v}$ proportional to $(-2\bar{K}\hat{\omega}_1/j)\,\tilde{S}_j$ as well as the contribution proportional to $2\bar{K}\,\tilde{C}_j\,\omega_1$.

At first sight we can only deal with the complication of the $(p/r)^2$ factor by multiplying each S_j by this factor, interpreted as $(1 + e\cos v)^2$, to yield a set of terms in S_{j-2}, S_{j-1}, S_j, S_{j+1} and S_{j+2}, and applying the general procedure to the individual terms. This is exactly what we have to do with the v and ω contributions to $da/d\bar{v}$, but for the $\hat{\omega}$ contribution, which we are currently concerned with, we can escape the complication by making use of a known

expression for a_1 directly. The argument of the last paragraph effectively shows (on multiplying through by $-j$) that to each term of a_1 in the form of a multiple of a particular $\tilde{C}_j$ there corresponds a carry-over contribution to $da/d\bar{v}$ consisting of the same multiple of $2\bar{K}^2 \hat{\omega}_1 \tilde{S}_j$. (The same principle applies to the analysis for every element, ζ , of course; when a multiple of $\tilde{S}_j$, as opposed to $\tilde{C}_j$, is involved in ζ_1 , arising from the appearance of $\tilde{S}_j/j$ in the first-order integration of $\tilde{C}_j$, then the second-order carry-over to $d\zeta/d\bar{v}$ is that same multiple of $-2\bar{K}^2 \hat{\omega}_1 \tilde{C}_j$.) The important point is that the factor $2\bar{K}^2 \hat{\omega}_1$ is independent of j , and this means that the argument is not affected by an unexpanded power of $(\bar{p}/\bar{r})$ in a_1 ; indeed, we can take the $\hat{\omega}$ contribution (to $da/d\bar{v}$) from the simplest possible expression for a_1 , *viz* (156). (The general principle, being independent of j , applies even when $j = 0$!)

With a self-explanatory notation, we may now list the six contributions to the second-order component of da/dv as follows, dropping all bars and tildes, even from $\bar{v}$ (it is remarked again that the D_a contribution covers the variation in n as well as a):

$$D_a(da/dv) = \frac{5}{4} K^2 q^{-2} (p/r)^2 Q_a a_1 , \tag{203}$$

$$D_j(da/dv) = -K^2 aq^{-2} (p/r)^2 sc(5eS_3 + 4S_2 - eS_1 - 6e \sin v) i_1 , \tag{204}$$

$$D_e(da/dv) = -\tfrac{1}{2}K^2 aq^{-2} (p/r) \left\{ Q_a \left[9eq^{-2}(p/r) + 4 \cos v \right] + (p/r) Q_{ae} \right\} e_1 , \tag{205}$$

where
$$Q_{ae} = 5fS_3 - fS_1 + 4h \sin v , \tag{206}$$

$$D_v(da/dv) = \tfrac{1}{2}K^2 aq^{-2} (p/r) \left\{ 4eQ_a \sin v - (p/r) Q_{av} \right\} v_1 , \tag{207}$$

where
$$Q_{av} = f(15eC_3 + 8C_2 - eC_1) + 4eh \cos v , \tag{208}$$

$$D_\omega(da/dv) = -K^2 aq^{-2} (p/r)^2 f(5eC_3 + 4C_2 - eC_1)\omega_1 \tag{209}$$

and

$$D_{\dot{\omega}}(da/dv) = 4K^2 aq^{-2} (p/r)^3 fg S_2 . \tag{210}$$

The six contributions are most conveniently dealt with by grouping into the four terms that contain Q_a as a common factor (the D_a-term, two D_e-terms and one D_v-term) and the five other terms (one associated with each of D_1, D_e, D_v, D_ω and $D_{\dot\omega}$). Seven of the nine terms contain $(p/r)^2$ (at least) as an explicit factor, the reason that only the first power of (p/r) appears in the other two terms being that these are the e_1 and v_1 components of what is really a term in $(p/r)_1$; but $(p/r)_1$ itself contains p/r as a factor, being given by (187); thus $(p/r)^2$ is an *overall* factor.

The combination of the two groups of terms may be denoted by $D_2(da/dv)$, and we find after some tedious algebra that

$$
\begin{aligned}
D_2(da/dv) = &- \frac{1}{48} K^2 a q^{-4} (p/r)^2 \times \\
&\times \Big[f^2 \Big\{ 15e^4\Sigma_8 + 102e^3\Sigma_7 + 6e^2[38,11]\Sigma_6 + 2e[103,128]\Sigma_5 + 4[16,86,3]\Sigma_4 \\
&\qquad + 2e[73,32]\Sigma_3 + 6e^2[6,1]\Sigma_2 - 6e^3\Sigma_1 - 3e^4\Sigma_0 \Big\} \\
&+ 8f \Big\{ 4e^4hS_6 + 26e^3hS_5 + e^2[3(30,-43),\ 4(5,-6)]S_4 \\
&\qquad + e[6(16,-23),\ (2,-9)]S_3 \\
&+ [24h,\ 6(2,-5),\ (34,-39)]S_2 - e[2(4,-3),\ -(22,-27)]S_1 \\
&\qquad + e^2[(2,3),\ -2(8,-9)]S_0 - 10e^3hS_{-1} - 2e^4hS_{-2} \Big\} \\
&+ 2\Big\{ e^3(8h^2 + 9f^2)(e \sin 4v + 6 \sin 3v) \\
&\qquad + 2e^2[8(8,-20,19),\ -(8,8,37)] \sin 2v \\
&\qquad + 2e[(64,-176,145),\ -4(2,-2,-11)] \sin v \Big\} \Big] .
\end{aligned}
\qquad ......(211)
$$

It is not obvious how we can integrate (211) without first expanding the factor $(p/r)^2$, but we are in the happy position of knowing what we expect the answer to be. Rather than attempt to integrate (211), therefore, we differentiate (199) with respect to v , using the fact that, eg,

$$\frac{d}{dv}\left\{(1+e\cos v)^3 C_j\right\} = -\tfrac{1}{2}(1+e\cos v)^2\left\{e(j+3)S_{j+1} + 2j\,S_j + e(j-3)S_{j-1}\right\} .$$

......(212)

We find, after some further tedious algebra, that $K^2\, da_2/dv$, derived in this way, is identical with $D_2(da/dv)$, as given by (211). This completes the check that the special and general methods yield the same result.

In sections 5.9 and 6.2 we shall need the expression for a_2 with $(p/r)^2$, rather than $(p/r)^3$, as a factor. This can be obtained at once from (199), of course, and will be given here for completeness. In section 5.5 we also need the expression with no p/r factor at all, but this is so lengthy that it seems best to suppress it. (We also suppress the version of a_2 that has a single factor p/r ; finally, it is easy to show that a further such factor cannot be removed from (199), which is on this basis the simplest possible expression for a_2 .)

The expression required in sections 5.9 and 6.2 is:

$$a_2 = \frac{1}{288}\, aq^{-4}(1+e\cos v)^2 \times$$

$$\times\ \Big[3f^2\left\{3e^4\Gamma_8 + 24e^3\Gamma_7 + 6e^2[11,3]\Gamma_6 + 4e[19,23]\Gamma_5 + 2[16,82,7]\Gamma_4\right.$$

$$\left. + 12e[9,5]\,\Gamma_3 + 2e^2[41,1]\Gamma_2 + 24e^3\Gamma_1 + 3e^4\Gamma_0\right\}$$

$$+ 24f\left\{e^4hC_6 + 8e^3hC_5 + e^2[(34,-49),-(6,-7)]C_4 + 4e[2(7,-10),-(0,1)]C_3\right.$$

$$+ [24h,2(18,-29),(10,-11)]C_2 + 4e[2(1,-3),3(4,-5)]C_1$$

$$\left. + e^2[(10,-21),3(6,-7)]C_0 + 8e^3hC_{-1} + e^4hC_{-2}\right\}$$

$$+ 2\left\{e^3(8h^2+9f^2)(e\cos 4v + 8\cos 3v)\right.$$

$$+ 2e^2[3(40,-104,103),-(8,24,-69)]\cos 2v$$

$$+ 4e[(112,-288,243),-3(0,16,-45)]\cos v$$

$$\left. + [8(20h^2+3f^2),6(24,-56,75),-3(8,8,-37)\right\}\Big] .$$

......(213)

5.3 Perturbation in η

The starting point is given by converting the exact equation (142) into an equation for $de/d\bar{v}$, making use of (39). The result is

$$\frac{de}{d\bar{v}} = -\tfrac{1}{8}K(n/\bar{n})(\bar{q}^3/q^3)\{(p/r)^3/(\bar{p}/\bar{r})^2\}Q_e , \tag{214}$$

where $$Q_e = f(5eS_4 + 14S_3 + 12eS_2 + 2S_1 - eS_0) + 4h(e \sin 2v + 2 \sin v) . \tag{215}$$

We proceed as in section 5.2, listing six contributions to the second-order component of $de/d\bar{v}$. With self-evident notation they are:

$$D_a(de/dv) = \tfrac{7}{16}K^2a^{-1}(p/r)Q_e a_1 , \tag{216}$$

$$D_i(de/dv) = -\tfrac{1}{4}K^2(p/r)sc(5eS_4 + 14S_3 + 12eS_2 + 2S_1 - eS_0 - 6e \sin 2v - 12 \sin v)i_1 , \tag{217}$$

$$D_e(de/dv) = -\tfrac{1}{8}K^2\{Q_e[7eq^{-2}(p/r) + 3 \cos v] + (p/r)Q_{ee}\}e_1 , \tag{218}$$

where $$Q_{ee} = f(5S_4 + 12S_2 - S_0) + 4h \sin 2v , \tag{219}$$

$$D_v(de/dv) = \tfrac{1}{8}K^2\{3eQ_e \sin v - (p/r)Q_{ev}\}v_1 , \tag{220}$$

where $$Q_{ev} = 2f(10eC_4 + 21C_3 + 12eC_2 + C_1) + 8h(e \cos 2v + \cos v) , \tag{221}$$

$$D_\omega(de/dv) = -\tfrac{1}{4}K^2(p/r)f(5eC_4 + 14C_3 + 12eC_2 + 2C_1 - eC_0)\omega_1 \tag{222}$$

and

$$D_{\dot\omega}(de/dv) = \tfrac{1}{12}K^2fg\{3e^2S_5 + 18eS_4 + (28 + 17e^2)S_3 + 60eS_2 + 3(4 + 11e^2)S_1 + 18eS_0 + 3e^2S_{-1}\} . \tag{223}$$

As in section 5.2, the six contributions may be grouped into four terms with Q_e as a common factor and five other terms. The first two Q_e terms (and only these) are found (after substitution for a_1 and e_1) to involve the factor q^{-2}, but their combination yields a cancelling factor q^2, so q^{-2} does not appear in the overall combination. Six of the nine terms contain the factor p/r and it can be extracted, as in section 5.2, from a combination of two of the others, but it does not occur in the $D_{\dot{\omega}}$ term, so p/r cannot be factored out of the overall combination. Some lengthy algebra eventually leads to:

$$
\begin{aligned}
D_2(de/dv) = \frac{1}{1536} K^2 e^{-1} \times \\
\times \Big[f^2 \Big\{ & 15e^4\Sigma_{10} + 162e^3\Sigma_9 + 8e^2[82,13]\Sigma_8 + 14e[84,53]\Sigma_7 + [784,1712,39]\Sigma_6 \\
& + 20e[54,1]\Sigma_5 - 8[56,58,53]\Sigma_4 - 12e[66,103]\Sigma_3 \\
& + [48,-624,-371]\Sigma_2 + 2e[36,-91]\Sigma_1 \quad - 18e^3\Sigma_{-1} - 3e^4\Sigma_{-2} \Big\} \\
+ 8f \Big\{ & 4e^4hS_8 + 42e^3hS_7 + e^2[172h,-(110,-141)]S_6 + 5e[64h,-3(38,-49)]S_5 \\
& + [224h -8(94,-123),-(338,-459)]S_4 \\
& - 2e[4(14,-23),5(78,-113)]S_3 - 2[80h,-2(26,9),3(10,-23)]S_2 \\
& - 6e[16(3,-4),-(98,-103)]S_1 - 8e^2q^2[14,-15]S_0 \\
& - e[48h,(158,-165)]S_{-1} - e^2[52h,3(18,-19)]S_{-2} \\
& \qquad - 18e^3hS_{-3} - 2e^4hS_{-4} \Big\} \\
+ 2 \Big\{ & e^3(8h^2 + 9f^2)(e \sin 6v + 10 \sin 5v) \\
& + 4e^2[2(40h^2 + 37f^2), -(24,-8,-31)] \sin 4v \\
& + 2e[36(8h^2 + 5f^2), -(296,-312,-165)] \sin 3v \\
& + [16(24h^2 + 7f^2), -16(72,-136,33), -(216,120,-551)] \sin 2v \\
& - 4e[2(88h^2 - 21f^2),(168,-88,-207)] \sin v \Big\} \Big] . \qquad (224)
\end{aligned}
$$

The term in S_0 is independent of v and represents our first source of long-period variation; the absence of a corresponding term in Σ_0 will be noted, and this applies to every element except σ (see section 5.8). We may write, in the notation of section 3.2,

$$e_{2,\ell p} = -\frac{1}{24}eq^2f(14-15f)S_0 , \tag{225}$$

the contribution to $\dot{e}_{\ell p}$ being given (*cf* (72)) by $K^2 n\, e_{2,\ell p}$. This leaves the short-period carry-over term given (*cf* (74)) by $K^2 e_{2,\ell p} m$. The method of integrating $e_{2,\ell p}$ with respect to time, to give the $J_2{}^2$ component of Δe , was discussed in section 3.5; see also Part 3.

The integration of the rest of (224) is immediate, except that we have to incorporate the right 'arbitrary constant'. This is naturally expressed via terms in Γ_0 and C_0 , together with a 'pure constant' (see the footnote in section 4.3) that is a function of e and i alone, the 'coefficients' being chosen to suit the derivation of r_2 in section 6.2. Anticipating the r_2 analysis, we derive (from the integration) the required result for e_2 , *viz*

$$e_2 = -\frac{1}{9216}e^{-1} \times$$

$$\times \Big[f^2\Big\{9e^4\Gamma_{10} + 108e^3\Gamma_9 + 6e^2[82,13]\Gamma_8 + 12e[84,53]\Gamma_7 + [784,1712,39]\Gamma_6$$

$$+ 24e[54,1]\Gamma_5 - 12[56,58,53]\Gamma_4 - 24e[66,103]\Gamma_3$$

$$+ 3[48,-624,-371]\Gamma_2 + 12e[36,-91]\Gamma_1 + 6e^2[66,-35]\Gamma_0$$

$$+ 108e^3\Gamma_{-1} + 9e^4\Gamma_{-2}\Big\}$$

$$+ 4f\Big\{6e^4hC_8 + 72e^3hC_7 + 2e^2[172h,-(110,-141)]C_6$$

$$+ 12e[64h,-3(38,-49)]C_5 + 3[224h,-8(94,-123), -(338,-459)]C_4$$

$$- 8e[4(14,-23),5(78,-113)]C_3 - 12[80h,-2(26,9), 3(10,-23)]C_2$$

$$- 72e[16(3,-4),-(98,-103)]C_1 +$$

$$+ [288h, 32(40,-33), (3002,-3255)]C_0$$

$$+ 12e[48h, (158,-165)]C_{-1} + 6e^2[52h, 3(18,-19)]C_{-2}$$

$$+ 72e^3hC_{-3} + 6e^4hC_{-4}\Big\}$$

$$+ 2\Big\{e^3(8h^2 + 9f^2)(e \cos 6v + 12 \cos 5v)$$

$$+ 6e^2[2(40h^2 + 37f^2), -(24,-8,-31)] \cos 4v$$

$$+ 4e[36(8h^2 + 5f^2), -(296,-312,-165)] \cos 3v$$

$$+ 3[16(24h^2 + 7f^2), -16(72,-136,33), -(216,120,-551)] \cos 2v$$

$$- 24e[2(88h^2 - 21f^2), (168,-88,-207)] \cos v$$

$$- 2[8(72h^2 + 29f^2), 6(264,-600,251), (632,-168,-1051)]\Big\}\Big] . \qquad \ldots\ldots(226)$$

5.4 Perturbation in i

The starting point, given by conversion of the exact equation (143), as in earlier sections, is

$$\frac{di}{d\bar{v}} = -K(n/\bar{n})(\bar{q}^3/q^3)\left\{(p/r)^3/(\bar{p}/\bar{r})^2\right\}Q_i , \qquad (227)$$

where $$Q_i = scS_2 . \qquad (228)$$

We proceed as usual, listing six contributions to the second-order component of $di/d\bar{v}$, *viz*

$$D_a(di/dv) = \tfrac{7}{2}K^2a^{-1}(p/r)Q_ia_1 , \qquad (229)$$

$$D_i(di/dv) = -K^2(p/r)(1 - 2f)S_2i_1 , \qquad (230)$$

$$D_e(di/dv) = -K^2\{7eq^{-2}(p/r) + 3\cos v\}Q_1e_1 , \quad (231)$$

$$D_v(di/dv) = K^2\{3eQ_1 \sin v - 2(p/r)\,scC_2\}v_1 , \quad (232)$$

$$D_\omega(di/dv) = -2K^2(p/r)\,scC_2\omega_1 \quad (233)$$

and

$$D_{\dot\omega}(di/dv) = \tfrac{2}{3}K^2\,scg\,(eS_3 + 3S_2 + 3eS_1) . \quad (234)$$

There are four terms with Q_1 as a common factor, and four other terms. As in section 5.3, the q^{-2} factor in the first e_1 term disappears when it is combined with the a_1 term. The overall combination is given by

$$\begin{aligned} D_2(di/d\bar{v}) = \frac{1}{48}K^2\,sc\{&e^2(2,5)\Sigma_6 + 10e(1,2)\Sigma_5 + 2[2(3,5),(4,3)]\Sigma_4 + 6e(3,2)\Sigma_3 \\ &+ e^2(6,1)\Sigma_2 \qquad - 2e^2(26,-33)S_4 - 20e(5,-7)S_3 \\ &+ 4[2(5,-3), -3(2,-3)]S_2 + 4e(31,-33)S_1 + 2e^2(14,-15)S_0 \\ &- 4e^2(3,-7)\sin 2v - 8e(3,-7)\sin v\} . \end{aligned} \quad (235)$$

The term in S_0 is, as in section 5.3, a source of long-period variation, given by

$$i_{2,\ell p} = \frac{1}{24}e^2\,sc(14 - 15f)S_0 , \quad (236)$$

with the inevitable short-period carry-over term. The remaining terms of (235) lead to the following formula, in which the term in C_0 and the 'pure constant' are chosen to suit the derivation of b_2 in section 6.4:

$$\begin{aligned} i_2 = -\frac{1}{288}\,sc\,\times \\ \times\{&e^2(2,5)\Gamma_6 + 12e(1,2)\Gamma_5 + 3[2(3,5),(4,3)]\Gamma_4 + 12e(3,2)\Gamma_3 + 3e^2(6,1)\Gamma_2 \\ &- 3e^2(26,-33)C_4 - 40e(5,-7)C_3 + 12[2(5,-3),-3(2,-3)]C_2 \\ &+ 24e(31,-33)C_1 + 4e^2(9,1)C_0 \qquad - 12e^2(3,-7)\cos 2v \\ &- 48e(3,-7)\cos v + 2[3(17,-25),-(90,-91)]\} . \end{aligned} \quad (237)$$

5.5 Check thus far by perturbations in p and $p\cos^2 i$

The absolute constancy of pc^2, for a disturbing function associated with only the zonal harmonics, has been referred to in sections 2.2 and 4.2. Now that we have second-order solutions for a, e and i, we can derive the corresponding solutions for p and c^2, and use their combination to check that pc^2 is indeed constant to second order.

Since there is no long-period variation in a, whilst the long-period variation in e is given by (225), it follows at once that the long-period variation in p is given by

$$p_{2.\ell p} = \frac{1}{12} pe^2 f(14 - 15f) S_0 \ . \qquad (238)$$

We obtain p_2 from the second-order identity

$$\bar{p} + \bar{K}p_1 + \bar{K}^2 p_2 = \left(\bar{a} + \bar{K}a_1 + \bar{K}^2 a_2\right)\left\{(1 - \bar{e}^2) - 2\bar{K}\bar{e}e_1 - \bar{K}^2(e_1^2 + 2\bar{e}e_2)\right\} , \qquad (239)$$

which gives (on dropping bars as usual)

$$p_2 = q^2 a_2 - 2aee_2 - e_1(ae_1 + 2ea_1) \ . \qquad (240)$$

From (147) and (148) we get

$$\begin{aligned} ae_1 + 2ea_1 = \frac{1}{48} aq^{-2} \Big[f\big\{ & 3e[1,3](eC_5 + 6C_4) + [28,133,19]C_3 + 12e[13,7]C_2 \\ & + 3[4,55,1]C_1 + 3e[1,3](6C_0 + eC_{-1}) \big\} \\ & + 4h\big\{[1,3](e^2\cos 3v + 6e\cos 2v + 3[4,1]\cos v) \\ & + 2e[13,7]\big\}\Big] \ . \qquad (241) \end{aligned}$$

We use (148) again, to obtain the product of (241) with e_1, then subtract the result from the combination of a_2 and e_2 specified by (240); the required form for a_2 was suppressed in section 5.2, it will be recalled, whilst e_2 is given by (226). We find, after the subtraction, that most of the terms have

cancelled out, and a factor q^4, needed to convert aq^{-2} in (241) into p, can be extracted from all that remain. The resulting expression for p_2 is:

$$p_2 = -\frac{1}{144}p\Big[f^2\{e^2\Gamma_6 - 3[2,5]\Gamma_4 - 48e\Gamma_3 - 33e^2\Gamma_2\}$$

$$- f\{3e^2(26,-33)C_4 + 8e(32,-41)C_3 + 24h[2,3]C_2 - 72e(8,-9)C_1$$

$$- 36e^2(8,-9)C_0\}$$

$$- 2\{6ef(4,-5)(e\cos 2v + 4\cos v) + [(8,-24,1),3(8,0,-15)]\}\Big] .$$

......(242)

From (236) and (238) it is immediately confirmed that there is no long-period variation in pc^2. Also, expansion of $\cos^2(\bar{i} + \bar{K}i_1 + \bar{K}^2 i_2)$ leads to

$$(c^2)_2 = (2f-1)i_1^2 - 2sci_2 , \tag{243}$$

from which it follows that

$$(pc^2)_2 = c^2p_2 - 2psci_2 + i_1\{p(2f-1)i_1 - 2scp_1\} . \tag{244}$$

From (149) and (172) we get

$$p(2f-1)i_1 - 2scp_1 = -\frac{1}{6}psc\{(1,2)(eC_3 + 3C_2 + 3eC_1) + (11,-18)\} . \tag{245}$$

Using (149) again, for the product of (245) with i_1, and adding the result to the required combination of p_2 and i_2, given by (242) and (237) respectively, we eventually obtain the final result we seek, *viz*

$$(pc^2)_2 = -\frac{1}{36}pc^2\{e^2f(63,-82)C_0 - [4(1,-6,2),(12,-50,13)]\} . \tag{246}$$

The absence of v-dependent terms in (246) completes our check of the constancy of pc^2 ; on inverting our standpoint, we have a powerful check on the expressions for a_2, e_2 and i_2 . (In practice, a number of errors were made in the e_2 analysis and were located via this check.) Expressions (175) and (246) provide the second-order relation between the mean and osculating values of pc^2 , connected (to second order) by

$$pc^2 = \bar{p}\bar{c}^2 + \bar{K}(pc^2)_1 + \bar{K}^2(pc^2)_2 . \tag{247}$$

In the light of (247) and the absolute constancy of pc^2 , (246) may appear as a paradoxical result, since the first-order variation of C_0 (interpreted as $\cos 2\tilde{\omega}$, with $\tilde{\omega}$ varying secularly) induces third-order variation in $\bar{K}^2(pc^2)_2$. But the second-order variation of $\bar{p}$ and $\bar{f}$ in $(pc^2)_1$, given by (175) with bars added, induces third-order variation in $\bar{K}(pc^2)_1$, and this may be regarded as cancelling the variation in $\bar{K}^2(pc^2)_2$. However, the real resolution of the paradox stems from the observation that, in the present analysis, it is meaningless anyway to regard $\bar{p}\bar{c}^2$, in (247), as contant to third order, since we have not carried the definitions of (semi-)mean elements beyond the second order.

5.6 Perturbation in Ω

The starting point, from the exact equation (144), is

$$\frac{d\Omega}{d\bar{v}} = - K(n/\bar{n})(\bar{q}^3/q^3)\{(p/r)^3/(\bar{p}/\bar{r})^2\}Q_\Omega , \tag{248}$$

where
$$Q_\Omega = c(1 - C_2) . \tag{249}$$

The six contributions to the second-order component of $d\Omega/d\bar{v}$ may be listed as usual, *viz*

$$D_a(d\Omega/dv) = \frac{7}{2} K^2 a^{-1}(p/r) Q_\Omega a_1 , \tag{250}$$

$$D_i(d\Omega/dv) = K^2(p/r)s(1 - C_2)i_1 , \tag{251}$$

$$D_e(d\Omega/dv) = - K^2\{7eq^{-2}(p/r) + 3 \cos v\} Q_\Omega e_1 , \tag{252}$$

$$D_v(d\Omega/dv) = K^2\{3eQ_\Omega \sin v - 2(p/r)cS_2\}v_1 , \tag{253}$$

$$D_\omega(d\Omega/dv) = -2K^2(p/r)cS_2\omega_1 \tag{254}$$

and

$$D_{\dot\omega}(d\Omega/dv) = -\tfrac{2}{3}K^2 c\,g(eC_3 + 3C_2 + 3eC_1) . \tag{255}$$

There are, as in section 5.4, four terms with Q_Ω as a common factor, and four other terms, the combination (free of the necessity for a q^{-2} factor, as usual) being expressible by:

$$\begin{aligned} D_2(d\Omega/dv) = &-\tfrac{1}{48}K^2c \times \\ &\times \{e^2(4,3)\Gamma_6 + 10e(2,1)\Gamma_5 + 2[4(3,1),(8,-1)]\Gamma_4 + 6e(5,-1)\Gamma_3 \\ &+ e^2(12,-5)\Gamma_2 - 52e^2(1,-1)C_4 - 8e(11,-8)C_3 \\ &+ 8[(8,-15),-(3,1)]C_2 + 8e(17,-30)C_1 \\ &+ 4e^2(7,-15)C_0 + 2e^2(4,5)\cos 2v \\ &+ 2e(2,1)\cos v + 2[4(1,-1),(4,5)]\} . \end{aligned}$$

......(256)

The final term in (256) is responsible for the first appearance of second-order secular variation; thus, it is given by

$$\hat\Omega_2 = -\tfrac{1}{24}c[4(1-f) + e^2(4+5f)] . \tag{257}$$

The term in C_0 in (256) leads, as in earlier sections, to the long-period variation given by

$$\Omega_{2,\ell p} = -\tfrac{1}{12}e^2c(7-15f)C_0 . \tag{258}$$

The remaining terms of (256) lead to the following formula for Ω_2, the 'constant' term in S_0 being chosen to suit the derivation of b_2 in section 6.4:

$$\Omega_2 = -\frac{1}{288} c \times$$

$$\times \Big\{ e^2(4,3)\Sigma_6 + 12e(2,1)\Sigma_5 + 3[4(3,1),(8,-1)]\Sigma_4 + 12e(6,-1)\Sigma_3$$

$$+ 3e^2(12,-5)\Sigma_2 \quad - 78e^2(1,-1)S_4 - 16e(11,-8)S_3$$

$$+ 24[(8,-15),-(3,1)]S_2 + 48e(17,-30)S_1 + 2e^2(18,-19)S_0$$

$$+ 6e^2(4,5) \sin 2v + 72e(2,1) \sin v \Big\} .$$

......(259)

5.7 Perturbation in ω

The starting point, from the exact (145), is

$$\frac{d\omega}{dv} = \frac{1}{8} K e^{-1}(n/\bar{n})(\bar{q}^3/q^3)\left\{(p/r)^3/(\bar{p}/\bar{r})^2\right\}\Omega_\omega , \tag{260}$$

where

$$\Omega_\omega = f(5eC_4 + 14C_3 + 2eC_2 - 2C_1 + eC_0 + 4e)$$
$$- 4h(2eC_2 - e\cos 2v - 2\cos v - 3e) . \tag{261}$$

Equation (261) has a certain inconformity, in comparison with (215), similar to that remarked upon in the first-order analysis in relation to (153) (propagated from (145)) since we do not have the neat division in which the f-terms involve precisely the C_j and the h-terms involve precisely the $\cos jv$.

The six contributions to the second-order component of $d\omega/d\bar{v}$ may be listed as usual; thus

$$D_a(d\omega/dv) = -\frac{7}{16} K^2 a^{-1} e^{-1} (p/r) Q_\omega a_1 , \tag{262}$$

$$D_i(d\omega/dv) = \frac{1}{4} K^2 e^{-1} (p/r)\, s\, c\, [5eC_4 + 14C_3 + 14eC_2 - 2C_1 + eC_0 - 2(3e \cos 2v + 6 \cos v + 7e)]\, i_1 , \tag{263}$$

$$D_e(d\omega/dv) = \frac{1}{8} K^2 e^{-1} \{ Q_\omega [7eq^{-2}(p/r) + 3 \cos v] - 2e^{-1}(p/r) Q_{\omega e} \} e_1 , \tag{264}$$

where

$$Q_{\omega e} = f(7C_3 - C_1) + 4h \cos v , \tag{265}$$

$$D_v(d\omega/dv) = -\frac{1}{8} K^2 e^{-1} \{ 3eQ_\omega \sin v + 2(p/r) Q_{\omega v} \} v_1 , \tag{266}$$

where $Q_{\omega v} = f(10eS_4 + 21S_3 + 2eS_2 - S_1) + 4h(e \sin 2v + \sin v - 2eS_2)$, (267)

$$D_\omega(d\omega/dv) = -\frac{1}{4} K^2 e^{-1} (p/r) \{ f(5eS_4 + 14S_3 + 2eS_2 - 2S_1 + eS_0) - 8ehS_2 \} \omega_1 \tag{268}$$

and

$$D_{\dot\omega}(d\omega/dv) = -\frac{1}{12} K^2 e^{-1} g \{ 3e^2 fC_5 + 18efC_4 + [28f,-(8,-19)]C_3 - 12e(2,-5)C_2 - 3[4f,(8,-15)]C_1 - 18efC_0 - 3e^2 fC_{-1} \} \tag{269}$$

There are four terms with Q_ω as a common factor, and five other terms, the combination (free of q^{-2} as usual) being expressible by:

$$D_2(d\omega/dv) = -\frac{1}{768} K^2 e^{-2} \Big[ef^2 \{ 15e^3\Gamma_{10} + 162e^2\Gamma_9 + 8e[82,15]\Gamma_8 + 14[84,67]\Gamma_7 \} + [784f^2, 2448f^2, -(64,16,-277)]\Gamma_6 + 20e[106f^2,-(16,0,-69)]\Gamma_5 - 8e^2[2(24,-4,-103),(32,-20,-23)]\Gamma_4 - 12e[18f^2,(48,-32,-35)]\Gamma_3 - [48f^2,208f^2,(192,-176,-9)]\Gamma_2 - ef^2 \{ 2[36,19]\Gamma_1 - 18e^2\Gamma_{-1} - 3e^3\Gamma_{-2} \} +$$

$$+ 8\{4e^4hfC_8 + 42e^3hfC_7 + e^2f[172h,-3(14,-17)]C_6$$

$$+ 5ef[64h,-(34,-39)]C_5 + [224hf,-12f(2,3),$$

$$(104,-242,159)]C_4$$

$$+ 8e[2f(21,-31),(22,-17,-5)]C_3 - 4e^2[2(16,-138,141),$$

$$-(12,22,-33)]C_2$$

$$+ 8e[f(4,-3),-(34,-173,150)]C_1 + 2e^2[2f(14,-15),$$

$$-(28,-158,135)]C_0$$

$$+ ef[48h,(110,-129)]C_{-1} + e^2f[52h,(34,-39)]C_{-2}$$

$$+ 18e^3hfC_{-3} + 2e^4hfC_{-4}\}$$

$$+ 2\{e^3(8h^2 + 9f^2)(e \cos 6v + 10 \cos 5v)$$

$$+ 4e^2[2(40h^2 + 37f^2),-(8,8,-37)] \cos 4v$$

$$+ 6e[12(8h^2 + 5f^2),-(24,24,-137)] \cos 3v$$

$$+ [16(24h^2 + 7f^2),-32(4,8,-41),-(264,-136,-301)] \cos 2v$$

$$+ 4e[2(8h^2 + 43f^2),-(256,-400,9)] \cos v$$

$$- 4e^2[2(64,-180,95),(56,-36,-45)]\}\Big] .$$

$$\ldots\ldots(270)$$

The final term in (270) represents, as in section 5.6, the secular variation, which is thus given by

$$\hat{\omega}_2 = \frac{1}{96}[2(64 - 180f + 95f^2) + e^2(56 - 36f - 45f^2)] . \qquad (271)$$

The term in C_0 leads, as usual, to the long-period variation given by

$$\omega_{2,\ell p} = -\frac{1}{48}[2f(14 - 15f) - e^2(28 - 158f + 135f^2)]C_0 . \qquad (272)$$

The remaining terms of (270) lead to the following formula for ω_2 , the 'constant' terms in Σ_0 and S_0 being chosen to suit the derivation of w_2 in section 6.4:

$$\omega_2 = -\frac{1}{4608}e^{-2} \times$$

$$\times \Big[3ef^2\{3e^3\Sigma_{10} + 36e^2\Sigma_9 + 2e[82,15]\Sigma_8 + 4[84,67]\Sigma_7\}$$

$$+ [784f^2,2448f^2,-(64,16,-277)]\Sigma_6 + 24e[106f^2,-(16,0,-69)]\Sigma_5$$

$$- 12e^2[2(24,-4,-103),(32,-20,-23)]\Sigma_4 - 24e[18f^2,(48,-32,-35)]\Sigma_3$$

$$- 3[48f^2, 208f^2, (192,-176,-9)]\Sigma_2$$

$$- 3ef^2\{4[36,19]\Sigma_1 + 2e[66,-1]\Sigma_0 + 36e^2\Sigma_{-1} + 3e^3\Sigma_{-2}\}$$

$$+ 8ef\{3e^3hS_8 + 36e^2hS_7 + e[172h,-3(14,-17)]S_6 + 6[64h,-(34,-39)]S_5\}$$

$$+ 4\{3[224hf,-12f(2,3),(104,-242,159)]S_4$$

$$+ 32e[2f(21,-31),(22,-17,-5)]S_3 - 24e^2[2(16,-138,141), -(12,22,-33)]S_2$$

$$+ 96e[f(4,-3),-(34,-173,150)]S_1 - [288hf,16f(116,-147), (144,-418,311)]S_0\}$$

$$- 24ef\{2[48h,(110,-129)]S_{-1} + e[52h,(34,-39)]S_{-2} + 12e^2hS_{-3} + e^3hS_{-4}\}$$

$$+ 2\{e^3(8h^2 + 9f^2)(e \sin 6v + 12 \sin 5v)$$

$$+ 6e^2[2(40h^2 + 37f^2),-(8,8,-37)] \sin 4v$$

$$+ 12e[12(8h^2 + 5f^2),-(24,24,-137)] \sin 3v$$

$$+ 3[16(24h^2 + 7f^2),-32(4,8,-41),-(264,-136,-301)] \sin 2v$$

$$+ 24e[2(8h^2 + 43f^2)-(256,-400,9)] \sin v\}\Big] .$$

$$\ldots\ldots(273)$$

5.8 Perturbation in σ

The starting point, from the exact equation (146), is

$$\frac{d\sigma}{d\bar{v}} = -\frac{1}{3}Ke^{-1}(n/\bar{n})(\bar{q}^3/q^2)\{(p/r)^3/(\bar{p}/\bar{r})^2\}Q_\sigma , \qquad (274)$$

where $Q_\sigma = f(5eC_4 + 14C_3 - 18eC_2 - 2C_1 + eC_0) + 4h(e\cos 2v + 2\cos v - 3e)$.

......(275)

The six contributions to the second-order component of $d\sigma/d\bar{v}$ may be listed as usual; thus

$$D_a(d\sigma/dv) = \frac{7}{16}K^2a^{-1}e^{-1}q(p/r)Q_\sigma a_1 , \qquad (276)$$

$$D_i(d\sigma/dv) = -\frac{1}{4}K^2e^{-1}q(p/r)\,s\,c\,[5eC_4 + 14C_3 - 18eC_2 - 2C_1 + eC_0$$
$$- 6(e\cos 2v + 2\cos v - 3e)]\,i_1 , \qquad (277)$$

$$D_e(d\sigma/dv) = -\frac{1}{8}K^2e^{-1}q\{Q_\sigma[6eq^{-2}(p/r) + 3\cos v] - 2e^{-1}(p/r)Q_{\sigma e}\}e_1 , \qquad (278)$$

where $\qquad Q_{\sigma e} = f(7C_3 - C_1) + 4h\cos v \quad (= Q_{\omega e}) , \qquad (279)$

$$D_v(d\sigma/dv) = \frac{1}{8}K^2e^{-1}q\{3eQ_\sigma \sin v + 2(p/r)Q_{\sigma v}\}v_1 , \qquad (280)$$

where $\qquad Q_{\sigma v} = f(10eS_4 + 21S_3 - 18eS_2 - S_1) + 4h(e\sin 2v + \sin v) , \qquad (281)$

$$D_\omega(d\sigma/dv) = \frac{1}{4}K^2e^{-1}q(p/r)\,f\{5eS_4 + 14S_3 - 18eS_2 - 2S_1 + eS_0\}\omega_1 \qquad (282)$$

and

$$D_{\dot\omega}(d\sigma/dv) = \frac{1}{12}K^2e^{-1}qfg\{3e^2C_5 + 18eC_4 + [28,-13]C_3 - 36eC_2$$
$$- 3[4,17]C_1 - 18eC_0 - 3e^2C_{-1}\} . \qquad (283)$$

There are four terms with Q_σ as a common factor, and five other terms, but (because the first D_e term has coefficient 6 instead of the usual 7) it is

not possible this time to extract the factor q^2 when this term is combined with the D_a term. The overall combination is expressible by:

$$D_2(d\sigma/dv) = \frac{1}{1536} K^2 e^{-2} q^{-1} \times$$

$$\times \Big[f^2 \{ 15e^4[2,-1]\Gamma_{10} + 162e^3[2,-1]\Gamma_9 + 2e^2[656,-208,-97]\Gamma_8$$

$$+ 2e[1176,350,-737]\Gamma_7 + [1568,4112,-3814,-21]\Gamma_6$$

$$+ 4e[1060,-704,-203]\Gamma_5 + 60e^2[16,-32,-5]\Gamma_4$$

$$- 36e[12,8,35]\Gamma_3 - [96,368,350,473]\Gamma_2 - 2e[72,2,121]\Gamma_1$$

$$- 18e^6\Gamma_0 + 18e^3[2,-1]\Gamma_{-1} + 3e^4[2,-1]\Gamma_{-2} \}$$

$$+ 8f \{ 4e^4 h[2,-1]C_8 + 42e^3 h[2,-1]C_7$$

$$+ e^2[344h,-8(32,-45),(86,-105)]C_6$$

$$+ e[640h,-30(22,-29),(382,-453)]C_5$$

$$+ [448h,-8(34,-33),2(298,-327),-(418,-531)]C_4$$

$$+ 2e[16(21,-31),4(60,-61),-(582,-749)]C_3$$

$$+ 2e^2[16(17,-21),-4(118,-147),-(10,-63)]C_2$$

$$+ 2e[8(4,-3),-8(95,-111),(506,-531)]C_1$$

$$+ 4e^2[2(14,-15),-4(44,-51),(106,-111)]C_0$$

$$+ e[96h,2(86,-93),-9(26,-31)]C_{-1}$$

$$+ e^2[104h,16,-3(22,-25)]C_{-2}$$

$$+ 18e^3 h[2,-1]C_{-3} + 2e^4 h[2,-1]C_{-4} \} +$$

$$+ 2\{e^3(8h^2 + 9f^2)[2,-1](e \cos 6v + 10 \cos 5v)$$

$$+ 2e^2[8(40h^2 + 37f^2),-8(24,-56,45),(24,56,-143)] \cos 4v$$

$$+ 6e[24(8h^2 + 5f^2),-2(72,-120,1),(40,72,-253)] \cos 3v$$

$$+ [32(24h^2 + 7f^2),-16(40,-40,-103),$$

$$2(312,88,-1331),-(504,-744,101)] \cos 2v$$

$$+ 4e[4(8h^2 + 43f^2),4(76,-156,-17),$$

$$-(488,-1176,689)] \cos v$$

$$+ 2e^2[16(16,-56,31),-16(16,-88,95),-(280,-328,-79)]\}\Big] . \qquad \ldots\ldots(284)$$

The final term at once yields (with the usual notation) $\hat{\sigma}_2$, and the long-period variation is given by the terms in Γ_0 and C_0 . However, there is little point in giving the explicit expressions, or in giving the formula for the short-period σ_2 , since σ (from its definition in section 2.1) is merely an intermediate element that has to be analysed as part of the analysis for M . The existence of a non-zero long-period term in Γ_0 , in addition to the usual term in C_0 , is another aspect of the ephemerality of σ , and it will be found in sections 5.9 and 5.10 that in the M-analysis the Γ_0 term is cancelled by an equal and opposite term from the quantity $\int$ defined by (3).

5.9 Perturbations in n and $\int$

The second-order perturbation in n , like the first-order perturbation, is purely short-periodic and may be obtained from the perturbation in a by use of the appropriate version of Kepler's third law for mean elements. The generalized second-order form of this law, with (non-dimensional) off-sets $\hat{\mu}_1$ and $\hat{\mu}_2$ to suit the definitions of $\bar{n}$ and $\bar{a}$, may be written

$$\bar{n}^2\bar{a}^3 = \mu\left(1 + \bar{K}\hat{\mu}_1 + \bar{K}^2\hat{\mu}_2\right) , \qquad (285)$$

but we have seen in section 4.1 that the natural first-order definition of $\bar{a}$ and $\bar{n}$ leads to the taking of $\hat{\mu}_1$ to be zero. There is no reason to expect $\hat{\mu}_2$ to be zero as well, however, and it will appear in section 5.10 that, with $\bar{a}$

defined to be identical with the exactly constant a', and $\bar{n}$ chosen to second order such that the secular variation in M vanishes,

$$\hat{\mu}_2 = -\frac{1}{24}q^3(8 - 8f - 5f^2) . \tag{286}$$

If we write, for convenience, $\hat{a}_j$ and $\hat{n}_j$ for $a_j/\bar{a}$ and $n_j/\bar{n}$ ($j = 1$ and 2), it follows that

$$n^2a^3 = \bar{n}^2\bar{a}^3\left(1 + \bar{K}\hat{a}_1 + \bar{K}^2\hat{a}_2\right)^3\left(1 + \bar{K}\hat{n}_1 + \bar{K}^2\hat{n}_2\right)^2 . \tag{287}$$

From (1), (285) and (287), we have at once that

$$\hat{\mu}_1 = -(3\hat{a}_1 + 2\hat{n}_1) \tag{288}$$

and

$$\hat{\mu}_2 = 6\hat{a}_1^{\,2} + 6\hat{a}_1\hat{n}_1 + 3\hat{n}_1^{\,2} - 3\hat{a}_2 - 2\hat{n}_2 . \tag{289}$$

Equation (288) just leads to (159), on the basis (from section 4.1) that $\hat{\mu}_1 = 0$; the new material is in (289), which with $\hat{n}_1$ set to $-\frac{3}{2}\hat{a}_1$ yields

$$\hat{n}_2 + \frac{1}{2}\hat{\mu}_2 = \frac{3}{8}\left(5\hat{a}_1^{\,2} - 4\hat{a}_2\right) . \tag{290}$$

The appropriate expression for a_1 is given by (158). When this is squared and combined with the expression for $\hat{a}_2$ given by (213), then for $\hat{\mu}_2$ given by (286) it is eventually found that

$$n_2 = \frac{1}{768}nq^{-4}(1 + e\cos v)^2 \times$$

$$\times\ \Big[3f^2\big\{3e^4\Gamma_8 + 24e^3\Gamma_7 + 12e^2[8,-1]\Gamma_6 + 8e[22,-1]\Gamma_5 + 2[56,32,17]\Gamma_4$$

$$+ 24e[2,5]\Gamma_3 + 4e^2[8,13]\Gamma_2 + 24e^3\Gamma_1 + 3e^4\Gamma_0\big\} +$$

$$+ 24f\{e^4hC_6 + 8e^3hC_5 - 2e^2[8(1,-1),-(22,-29)]C_4$$

$$- 4e[4(4,-5),-(30,-41)]C_3 - [16h,-32(3,-4),(1[illegible])]C_2$$

$$+ 4e[4(8,-9),-3(6,-5)]C_1 + 2e^2[8(5,-6),-(26,-27)]C_0$$

$$+ 8e^3hC_{-1} + e^4hC_{-2}\}$$

$$+ 2\{e^3(8h^2 + 9f^2)(e \cos 4v + 8 \cos 3v)$$

$$- 4e^2[96f(1,-2),-(56,-72,-3)] \cos 2v$$

$$- 8e[2(32,-48,-27),-3(40,-88,45)] \cos v$$

$$- [8(40h^2 - 33f^2),-96(4,-16,15),-3(72,-88,-13)]\}]$$

$$+ \frac{1}{48} nq^3(8 - 8f - 5f^2) \; .$$

$$\ldots\ldots(291)$$

The second-order perturbation in M, to be derived in section 5.10, is given by the combination of the perturbation in σ with the perturbation in $\int$, the integral of n; since (in section 5.8) we only took the analysis as far as $D_2(d\sigma/dv)$, however, what we need here is $D_2(d\int/dv)$, to combine with $D_2(d\sigma/dv)$ to give $D_2(dM/dv)$.

Now (291) displays n_2 as a sum of many terms, all with $(1 + e \cos v)^2$ as a factor, together with a final term that emanates from $\hat{\mu}_2$. Apart from a complication to be dealt with in the next paragraph, the 'sum of many terms' gives an immediate contribution to $D_2(d\int/dv)$, since (with bars omitted as usual)

$$\frac{d\int_2}{d\bar{v}} = \frac{n_2}{nW} = \frac{n_2 q^3}{n(1 + e \cos v)^2} \tag{292}$$

by (39) and (17). Thus the factor $(1 + e \cos v)^2$ in (291) is conveniently cancelled and the required contribution from the sum of many terms is given on replacing $nq^{-4}(1 + e \cos v)^2$ by K^2q^{-1}. This just leaves the 'final term', which contributes $\frac{1}{48} K^2 q^3 W^{-1}(8 - 8f - 5f^2)$, since $\partial v/\partial M = W$.

The complication in the 'sum of many terms', referred to in the last paragraph, arises because $D_2(d\int/dv)$ has to cover some carry-over terms from n_1 as well as the direct terms associated with n_2 . The carry-over terms arise from the first-order secular variation in ω , and constitute a final manifestations of the '$D_{\dot\omega}$ source' of terms introduced in section 5.2 and encountered in all the subsequent sections. Here, the origin of the terms is the first-order evaluation of $\int_1$ by v-integration of (160). As in section 5.2, to compensate for the neglect of the term $(-2\bar{K}\tilde{\omega}_1/j^2)\tilde{S}_j$, when replacing each integral of $\tilde{C}_j$ by $\tilde{S}_j/j$, we need a contribution to $D_2(d\int/dv)$ given by $\frac{1}{4}qf(4-5f)(eC_3+3C_2+3eC_1)$, and the effect of this on the conversion of (291) into the expression required for $D_2(d\int/dv)$ is to replace the terms in C_3, C_2 and C_1 (within the '24f bracket') by

$$-4e[2(4,-5),-(22,-31)]C_3 + [16(5,-6),-8f,-(10,1)]C_2$$
$$+ 4e[2(28,-33),-3(14,-15)]C_1 \ .$$

5.10 Perturbation in M

The expression for $D_2(dM/dv)$ is given at once by the combination of (284) with the expression for $D_2(d\int/dv)$ obtained from (291) as indicated in the last section. The result is:

$$D_2(dM/dv) = \frac{1}{1536}K^2e^{-2}q^{-1} \times$$
$$\times \Big[f^2\{15e^4[2,-1]\Gamma_{10} + 162e^3[2,-1]\Gamma_9 + 16e^2[82,-26,-11]\Gamma_8$$
$$+ 14e[168,50,-95]\Gamma_7 + [1568,4112,-3238,-93]\Gamma_6$$
$$+ 20e[212,-88,-43]\Gamma_5 + 96e^2[17,-16,-1]\Gamma_4$$
$$- 108e[4,0,5]\Gamma_3 - [96,368,158,161]\Gamma_2 - 2e[72,2,49]\Gamma_1$$
$$+ 18e^3[2,-1]\Gamma_{-1} + 3e^4[2,-1]\Gamma_{-2}\}$$
$$+ 8f\{4e^4h[2,-1]C_8 + 42e^3h[2,-1]C_7$$
$$+ 2e^2[172h,-4(32,-45),(45,-57)]C_6$$
$$+ [448h,-8(34,-33),2(250,-279),-(154,-183)]C_4 +$$

$$
\begin{aligned}
&+ 2e[16(21,-31),4(36,-31),-(318,-377)]C_3 \\
&+ 8e^2[4(32,-39),-(118,-141),-5(2,-3)]C_2 \\
&+ 2e[8(4,-3),-8(11,-12),(2,9)]C_1 \\
&+ 8e^2q^4(14,-15)C_0 + e[96h,2(86,-93),-3(62,-69)]C_{-1} \\
&+ 2e^2[52h,8,-3(10,-11)]C_{-2} + 18e^3h[2,-1)C_{-3} \\
&\qquad\qquad + 2e^4h[2,-1]C_{-4}\Big\} \\
&+ 2\Big\{e^3(8h^2 + 9f^2)[2,-1](e\cos 6v + 10\cos 5v) \\
&\quad + 8e^2[2(40h^2 + 37f^2),-2(24,-56,45),(8,8,-29)]\cos 4v \\
&\quad + 2e[72(8h^2 + 5f^2),-6(72,-120,1),(184,24,-543)]\cos 3v \\
&\quad + [32(24h^2 + 7f^2),-16(40,-40,-103),2(312,-296,-563), \\
&\qquad\qquad -(56,-168,125)]\cos 2v \\
&\quad + 4e[4(8h^2 + 43f^2),4(12,-60,37),-(8,-120,149)]\cos v \\
&\qquad\qquad - 16e^2q^4(8,-8,-5)\Big\}\Big] \\
&\qquad\qquad + \frac{1}{48}K^2q^3W^{-1}(8 - 8f - 5f^2) \;. \qquad \ldots\ldots(293)
\end{aligned}
$$

The basis for the choice of $\hat{\mu}_2$ defined by (286), which determined the final terms of both (291) and (293), should now be apparent, since the combination of the final term in (293) with the last of the preceding collection of terms is $\frac{1}{48}K^2q^3(8 - 8f - 5f^2)(W^{-1} - 1)$, the mean value of which (with respect to v) is zero by (39). The penultimate term of (293) in fact defines $\hat{M}_2$, *viz*

$$
\hat{M}_2 = -\frac{1}{48}q^3(8 - 8f - 5f^2) \;, \tag{294}
$$

and $\hat{\mu}_2$ was set to $2\hat{M}_2$ to allow for the resulting secular effect, as anticipated in the last section and (before that) in section 3.1. Though our choice of $\hat{\mu}_2$ in principle allows for $\hat{M}_2$, making $\bar{n}$ and $\dot{M}_{sec}$ identical as required, we must not overlook the short-period carry-over effect of $\hat{M}_2$. This can be allowed for by applying (64) with $\zeta = M$.

As forecast in section 5.8, (293) contains no term in Γ_0. The term in C_0 leads, as usual, to the long-period variation given by

$$M_{2,\ell p} = \frac{1}{24} q^3 f(14 - 15f) C_0 . \tag{295}$$

The remaining terms of (293) lead to the following formula for M_2, the 'constant' terms in Σ_0 and S_0 being chosen to suit the derivation of r_2 in section 6.2:

$$\begin{aligned}
M_2 = \frac{1}{9216} e^{-2} q^{-1} \Big[f^2 \Big\{ & 9e^4[2,-1]\Sigma_{10} + 108e^3[2,-1]\Sigma_9 + 12e^2[82,-26,-11]\Sigma_8 \\
& + 12e[168,50,-95]\Sigma_7 + [1568,4112,-3238,-93]\Sigma_6 \\
& + 24e[212,-88,-43]\Sigma_5 + 144e^2[17,-16,-1]\Sigma_4 - 216e[4,0,5]\Sigma_3 \\
& - 3[96,368,158,161]\Sigma_2 - 12e[72,2,49]\Sigma_1 - 12e^2[66,-34,13]\Sigma_0 \\
& \qquad - 108e^3[2,-1]\Sigma_{-1} - 9e^4[2,-1]\Sigma_{-2} \Big\} \\
+ 4f \Big\{ & 6e^4h[2,-1]S_8 + 72e^3h[2,-1]S_7 \\
& + 4e^2[172h,-4(32,-45),(46,-57)]S_6 \\
& + 12e[128h,-6(22,-29),(86,-105)]S_5 \\
& + 3[448h,-8(34,-33),2(250,-279),-(154,-183)]S_4 \\
& + 8e[16(21,-31),4(36,-31),-(318,-377)]S_3 \\
& + 48e^2[4(32,-39),-(118,-141),-5(2,-3)]S_2 \\
& + 24e[8(4,-3),-8(11,-12),(2,9)]S_1 -
\end{aligned}$$

$$- [576h, 16(214,-267), -2(802,-963), -(830,-861)]S_0$$

$$- 12e[96h, 2(86,-93), -3(62,-69)]S_{-1} - 12e^2[52h, 8, -3(10,-11)]S_{-2}$$

$$- 72e^3h[2,-1]S_{-3} - 6e^4h[2,-1]S_{-4}\Big\}$$

$$+ 2\Big\{e^3(8h^2 + 9f^2)[2,-1](e \sin 6v + 12 \sin 5v)$$

$$+ 12e^2[2(40h^2 + 37f^2), -2(24,-56,45), (8,8,-29)] \sin 4v$$

$$+ 4e[72(8h^2 + 5f^2), -6(72,-120,1), (184,24,-543)] \sin 3v$$

$$+ 3[32(24h^2 + 7f^2), -16(40,-40,-103), 2(312,-296,-563),$$

$$-(56,-168,125)] \sin 2v$$

$$+ 24e[4(8h^2 + 43f^2), 4(12,-60,37), -(8,-120,149)] \sin v\Big\}\Big] .$$

.......(296)

6 ADOPTED SOLUTION FOR $J_2{}^2$ PERTURBATIONS

6.1 Secular and long-period terms, plus short-period carry-over

Solution of the J_2 problem to second order amounts to the extension of the first-order solution of section 4 by additional terms. The terms expressing the long-term variation in the mean elements are compact and small in number and, since they are not easily to be found among the much longer expressions of section 5, it is convenient to repeat them here. Though the $\zeta_{2,\ell p}$ are actually all non-singular, propagation of the ζ will nevertheless be via $\psi_{2,\ell p}$ and $L_{2,\ell p}$ (rather than $\omega_{2,\ell p}$ and $M_{2,\ell p}$), as explained in section 3.5 and Part 3, so we include here also the expressions for these.

Each $\zeta_{2,\ell p}$ induces a short-period carry-over (*cf* (74)), and these are dealt with by combining them into components of the perturbations in the spherical coordinates, as indicated in section 3.2; the resulting expressions (components of δr, δb and δw) are included in the present section. The pure short-period components, representd by r_2, b_2 and w_2 , are derived in sections 6.2 to 6.4.

The only secular terms are in Ω and ω . They are given by $\bar{K}^2\bar{n}\hat{\Omega}_2 t$ and $\bar{K}^2\bar{n}\hat{\omega}_2 t$ where, to repeat (257) and (271) with bars added,

$$\hat{\Omega}_2 = -\frac{1}{24}\bar{c}[4(1-\bar{f}) + \bar{e}^2(4+5\bar{f})] \tag{297}$$

and

$$\hat{\omega}_2 = \frac{1}{96}[2(64 - 180\bar{f} + 95\bar{f}^2) + \bar{e}^2(56 - 36\bar{f} - 45\bar{f}^2)] \ . \tag{298}$$

A secular perturbation in M is avoided, essentially because, following (285) and (286), Kepler's third law is used in the modified form

$$\bar{n}^2\bar{a}^3 = \mu\left[1 - \frac{1}{24}\bar{K}^2\bar{q}^3(8 - 8\bar{f} - 5\bar{f}^2)\right] \ . \tag{299}$$

However, $\hat{M}_2$, given by (294), induces a second-order carry-over contribution to the semi-mean $\tilde{M}$, just as $\hat{\Omega}_2$ and $\hat{\omega}_2$ do, to $\tilde{\Omega}$ and $\tilde{\omega}$ respectively.

The long-period perturbations are given by the integration of the various $\bar{K}^2\bar{n}\zeta_{2,\ell p}$, following the methods indicated in section 3.5 and Part 3. The expressions for the $\zeta_{2,\ell p}$ are given by (225), (236), (258), (272) and (295), repeated here as

$$e_{2,\ell p} = -\frac{1}{24}\bar{e}\bar{q}^2\bar{f}(14 - 15\bar{f})\sin 2\bar{\omega} \ , \tag{300}$$

$$i_{2,\ell p} = \frac{1}{24}\bar{e}^2\bar{s}\bar{c}(14 - 15\bar{f})\sin 2\bar{\omega} \ , \tag{301}$$

$$\Omega_{2,\ell p} = -\frac{1}{12}\bar{e}^2\bar{c}(7 - 15\bar{f})\cos 2\bar{\omega} \ , \tag{302}$$

$$\omega_{2,\ell p} = -\frac{1}{48}\left[2\bar{f}(14 - 15\bar{f}) - \bar{e}^2(28 - 158\bar{f} + 135\bar{f}^2)\right]\cos 2\bar{\omega} \tag{303}$$

and

$$M_{2,\ell p} = \frac{1}{24}\bar{q}^3\bar{f}(14 - 15\bar{f})\cos 2\bar{\omega} \ . \tag{304}$$

From (302)-(304) we get the expressions to replace (303) and (304) in practice, *viz*

$$\Psi_{2,\ell p} = -\frac{1}{48}(2 + 5\bar{e}^2)\bar{f}(14 - 15\bar{f}) \cos 2\bar{\omega} \tag{305}$$

and

$$L_{2,\ell p} = -\frac{7}{48}\bar{e}^2\bar{q}\bar{f}(14 - 15\bar{f}) \cos 2\bar{\omega}\,. \tag{306}$$

The short-period effects induced by (300)-(304) are combined into spherical coordinate perturbations by use of (95)-(97). The results may be expressed, in a new but obvious notation (in which the carry-over r_{2co} *etc* will be added directly to the r_2 *etc* to be developed in sections 6.2 to 6.4), as

$$r_{2co} = \frac{1}{24} p e f (14 - 15f)\, m S_1 \,, \tag{307}$$

$$b_{2co} = -\frac{1}{48}\, e^2 s c m\, [15f\gamma_1 - (28 - 45f)c_{-1}] \tag{308}$$

and

$$w_{2co} = \frac{1}{48} e f (14 - 15f)\, m (eC_2 + 4C_1 - 4eC_0)\,. \tag{309}$$

(In equation (43) of Ref 4, the factor e^2 s c is missing in the terms of b_{2co} .)

6.2 Pure short-period perturbation in r

Since r is a function of the three elements a, e and M , with first-order partial derivatives given by (41), evaluation of r_2 in principle entails the combining of nine terms from a Taylor expansion, three of them involving a_2, e_2 and M_2 through first-order derivatives, whilst the other six involve the second-degree products of a_1, e_1 and M_1 through second-order derivatives. Thus the analysis is potentially much more complicated than for v_2 and b_2 (to follow in sections 6.3 and 6.4), each of which involves only a pair of orbital elements (not three) and hence only five Taylor terms (two plus three). A somewhat different approach was followed, therefore, 'tailored' rather than 'Taylored'; it is based on the use of eccentric anomaly, r being the simple function of a, e and E given by equation (78).

We need expressions for E_1 and E_2 , defined to represent pure short-period perturbations in the usual way, and these can be obtained from (75). We

replace e by $(\bar{e} + \bar{K}e_1 + \bar{K}^2 e_2)$ in this equation, and likewise for M and E, then expand $\sin E$ (and thence $e \sin E$) to $O(\bar{K}^2)$, and can thus identify expressions equivalent to M_1 and M_2. Rearrangement of these expressions, with algebraic simplification through the introduction of v where possible, gives us

$$E_1 = q^{-1} e_1 \sin v + q^{-2} M_1 (1 + e \cos v) , \tag{310}$$

where e, q and v are of course to be read as barred, and

$$E_2 = (a/r) M_2 + q^{-1} e_2 \sin v + \tfrac{1}{2}(a/r) E_1 [e_1 (\cos v + \cos E) - q^{-1} e M_1 \sin v] . \tag{311}$$

Equations (310) and (311) can now be used, as required, in the expressions obtained by expansion of (78). At the first-order level this expansion gives

$$r_1 = (r/a) a_1 - a e_1 \cos E - a e E_1 \sin E , \tag{312}$$

from which the use of (310) leads at once to a formula for r_1 that is equivalent to (95). At the second-order level the expansion yields

$$\begin{aligned} r_2 = {} & (r/a) a_2 - a e_2 \cos E + a e E_2 \sin E - a_1 (e_1 \cos E - e E_1 \sin E) \\ & + \tfrac{1}{2} a E_1 (2 e_1 \sin E + e E_1 \cos E) , \end{aligned} \tag{313}$$

from which the use of (310) and (311) leads to

$$\begin{aligned} r_2 = {} & (r/a) a_2 - a_1 (e_1 \cos v - e q^{-1} M_1 \sin v) \\ & + \tfrac{1}{4} a q^{-2} [e_1 \sin v + q^{-1} M_1 (1 + e \cos v)] \times \\ & \quad \times [e_1 (4 \sin v + e \sin 2v) + e q^{-1} M_1 (e + 2 \cos v + e \cos 2v)] \\ & + a e q^{-1} M_2 \sin v - a e_2 \cos v . \end{aligned} \tag{314}$$

The five terms, of varying complexity, in (314) have been presented in an order that reflects the way in which they are most naturally combined. To start with, we combine the first two terms. Since (r/a) is simply $q^2(1 + e\cos v)^{-1}$, the first term is given by (213) with the overall coefficient replaced by $\frac{1}{288}aq^{-2}(1 + e\cos v)$. Also, (148) and (164) give

$$e_1 \cos v - eq^{-1}M_1 \sin v = \frac{1}{12}\left\{f(3e^2C_4 + 12eC_3 + 2[5,4]C_2 + 12eC_1 + 3e^2C_0) + 4h(e^2\cos 2v + 4e\cos v + 3)\right\},$$

......(315)

the product of which with (158) yields an expression for the second term of (314), with $(1 + e\cos v)$ as a factor. To avoid another long expression here (and likewise at some other points of section 6) we give the result in a skeleton form in which only the 'end terms' are quoted; it is

$$-\frac{1}{288}aq^{-2}(1 + \cos v)\Big[3f^2\left\{3e^4\Gamma_8 + \dots + 3e^4\Gamma_0\right\} + 8fh\left\{3e^4C_6 + \dots + 3e^4C_{-2}\right\} + 2\left\{e^4[8h^2 + 9f^2]\cos 4v + \dots + 8h^2[12,22,1] + 3f^2[20,74,11]\right\}\Big] .$$

The first two terms of (314) can now be combined, whereupon a considerable simplification occurs since ten pairs of terms cancel out and a factor $4q^2$ can be extracted from the remaining terms. The result is worth expressing in full; thus

$$\begin{aligned}(r/a)a_2 - a_1(e_1 \cos v - eq^{-1}M_1 \sin v) = -\frac{1}{72}a(1 + e\cos v)\Big[&3f^2\left\{e^2\Gamma_6 + 3e\Gamma_5 + 2q^2\Gamma_4 - 5e\Gamma_3 - 3e^2\Gamma_2\right\} \\ &- 8f\left\{e^2(7,-9)C_4 + e(16,-21)C_3 + [2(2,-3),-(4,-3)]C_2 - e(20,-21)C_1 - e^2(11,-12)C_0\right\} \\ &- 2\left\{e^2(8,0,-15)\cos 2v + e(32,-48,-3)\cos v + [(16,-48,27),(8,0,-15)]\right\}\Big] .\end{aligned}$$

... ..(316)

Turning to the third term of (314), we find that

$$e_1 \sin v + q^{-1} M_1 (1 + e \cos v)$$

$$= -\frac{1}{48} e^{-1} \Big[f \big\{ 3e^2 S_5 + 6e[3,-1] S_4 + [28,-13] S_3 + 16eq^2 S_2 - 3[4,1] S_1$$

$$- 6e[3,-1] S_0 - 3e^2 S_{-1} \big\}$$

$$+ 4h \big\{ e^2 \sin 3v + 2e[3,-1] \sin 2v + [12,-7] \sin v \big\} \Big]$$

......(317)

and

$$e_1 (4 \sin v + e \sin 2v) + eq^{-1} M_1 (e + 2 \cos v + e \cos 2v)$$

$$= \frac{1}{48} \Big[f \big\{ 3e^2 S_6 + 6e[3,1] S_5 + 2[14,19] S_4 + 2e[25,11] S_3$$

$$- 48q^2 S_2 - 2e[41,-5] S_1 - 6[2,9] S_0 - 6e[3,1] S_{-1} - 3e^2 S_{-2} \big\}$$

$$+ 4h \big\{ e^2 \sin 4v + 2e[3,1] \sin 3v + 2[6,5] \sin 2v + 2e[11,1] \sin v \big\} \Big] .$$

......(318)

Multiplying the product of these two expressions by a further factor $\frac{1}{4} aq^{-2}$, we get the required 'third term'; in skeleton form the resulting expression is

$$\frac{1}{18432} ae^{-1} q^{-2} \Big[f^2 \big\{ 9e^4 \Gamma_{11} + \ldots + 9e^4 \Gamma_{-3} \big\} + 8hf \big\{ 3e^4 C_9 + \ldots + 3e^4 C_{-5} \big\}$$

$$+ 2 \big\{ e^4 (8h^2 + 5f^2) \cos 7v + \ldots - 8e[2(168h^2 + 73f^2), -(88h^2 - 197f^2),$$

$$-4(8h^2 + 25f^2)] \big\} \Big] .$$

This expression combines well with the fourth term of (314), given by multiplying the expression for M_2 , viz (296), by $aeq^{-1} \sin v$; the result of that multiplication, in skeleton form, is

$$
\begin{aligned}
-\frac{1}{18432}\,ae^{-1}q^{-2}\ \Big[f^2\{9e^4[2,-1]\Gamma_{11} + \ldots + 9e^4[2,-1]\Gamma_{-3}\} \\
+ 4f\{6he^4[2,-1]C_9 + \ldots + 6he^4[2,-1]C_{-5}\} \\
+ 2\{e^4[2,-1](8h^2 + 9f^2)\cos 7v + \ldots - 24e[4(8h^2 + 43f^2), 4(12,-60,37), \\
-(8,-120,149)]\}\Big].
\end{aligned}
$$

The virtue of combining the third and fourth terms of (314) is that a common factor q^2 emerges, to cancel the existing q^{-2} in both terms; the resulting expression (in full) is

$$
\begin{aligned}
-\frac{1}{18432}\,ae^{-1}\ \Big[f^2\{9e^4\Gamma_{11} + 108e^3\Gamma_{10} + 3e^2[164,29]\Gamma_9 + 24e[42,31]\Gamma_8 \\
+ [784,2204,-267]\Gamma_7 + 36e[64,-35]\Gamma_6 + [112,-1944,-325]\Gamma_5 \\
- 16e[98,25]\Gamma_4 - [528,-328,-347]\Gamma_3 - 12e[96,-55]\Gamma_2 \\
+ 9[16,-164,-19]\Gamma_1 + 24e[18,-41]\Gamma_0 + 3e^2[132,-67]\Gamma_{-1} \\
+ 108e^3\Gamma_{-2} + 9e^4\Gamma_{-3}\} \\
+ 4f\{6e^4hC_9 + 72e^3hC_8 + e^2[344h,-(214,-273)]C_7 \\
+ 24e[32h,-3(18,-23]C_6 + [672h,-4(478,-609),(558,-645)C_5 \\
+ 4e[8(10,-13),(798,-925)]C_4 - [288h,-16(436,-519), \\
-(2,-147)]C_3 \\
+ 16e[2(38,-41),-(138,-139)]C_2 - (672h,8(914,-1047), \\
(1198,-1413)]C_1 \\
- 36e[8(10,-13),(50,-59)]C_0 + [288h,4(398,-381), \\
5(102,-105)]C_{-1} \\
+ 24e[24h,(82,-87)C_{-2} + 3e^2[104h,(110,-117)]C_{-3} \\
+ 72e^3hC_{-4} + 6e^4hC_{-5}\} +
\end{aligned}
$$

$$+ 2\{e^4[8h^2 + 9f^2] \cos 7v + 12e^3[8h^2 + 9f^2] \cos 6v$$

$$+ e^2[12(40h^2 + 37f^2),-(136,-24,-213)] \cos 5v$$

$$+ 8e[18(8h^2 + 5f^2),-(136,-120,-123)] \cos 4v$$

$$+ [48(24h^2 + 7f^2),-12(248,-424,5),(232,-312,-81)] \cos 3v$$

$$- 4e[48(16h^2 - 9f^2),-(232,-696,351)] \cos 2v$$

$$- [16(72h^2 + 37f^2),-24(104,-152,-61),(104,-312,-97)] \cos v$$

$$+ 8e[10(24h^2 - 37f^2),(8,264,-275)]\}\Big] .$$

The terms in Γ_1, Γ_{-1}, C_1 and C_{-1} here are, of course, determined by the apparently arbitrary choice of 'constant' terms in Σ_0 and S_0 that was made in (296), but we are now close to seeing why the particular choice of coefficients was made.

We are left with only the last term of (314). Multiplying the expansion for e_2, *viz* (226), by $-a \cos v$, we obtain for this last term, quoting in skeleton form only,

$$-a e_2 \cos v = \frac{1}{18432} a e^{-1} \Big[f^2\{9e^4\Gamma_{11} + \ldots + 9e^4\Gamma_{-3}\}$$

$$+ 4f\{6he^4C_9 + \ldots + 6he^4C_{-5}\}$$

$$+ 2\{e^4(8h^2 + 9f^2) \cos 7v + \ldots - 24e[2(88h^2 - 21f^2),$$

$$(168,-88,-207)]\}\Big] . \qquad \ldots\ldots(319)$$

When the full version of (319) is combined with the long preceding expression, a tremendous simplification occurs, since no less than 20 pairs of terms cancel out and a factor 16e can be extracted from the remaining terms. The result, which represents the last three terms of (314), is worth giving in full, being

$$\frac{1}{1152}\,a\left[f^2\left\{24e^3\Gamma_7 + 120e^2\Gamma_6 + e[185,-17]\,\Gamma_5 + 16[5,-8]\Gamma_4 - e[181,131]\Gamma_3 - 264e^2\Gamma_2 - 72e^3\Gamma_1\right\}\right.$$

$$- 4f\left\{16e^3\,(7,-9)C_5 + 48e^2(10,-13)C_4 + e[(538,-717),(86,-147)]C_3 + 32[(10,-13),-2(6,-5)]\,C_2 - 48e[12(1,-1),5(1,-1)]C_1 - 48e^2(14,-15)C_0 - 16e^3(11,-12)C_{-1}\right\}$$

$$\left. - 16\left\{e^3(8,0,-15)\,\cos 3v + 3e^2(16,-16,-11)\,\cos 2v + 3e[16(2,-4,1),(8,0,-15)]\,\cos v + [(48,-144,77),8(4,0,-7)]\right\}\right]\,.$$

We can now make the grand combination of this last expression, which represents all but the first two terms of (314), with (316), which (on multiplying out the factor $1 + e\cos v$) compatibly represents the first two terms. In the process, another 12 pairs of terms cancel out and a common factor q^2 can be extracted, which has the effect of changing an external factor from 'a' into 'p' ; the final result is

$$r_2 = -\frac{1}{1152}\,p\left[f^2\left\{7e\Gamma_5 + 16\Gamma_4 - 11e\Gamma_3\right\} - 4f\left\{e(38 - 51f)C_3 - 32(6 - 7f)C_2\right\} + 16(16h^2 - 13f^2)\right]\,. \tag{320}$$

As one of the definitive formulae of the paper, this has been written without use of the bracket conventions; its brevity, in comparison with the individual components of (314), is striking, and it is likely that there is some physical significance in this compression.

It should now be clear how the coefficients of Γ_0 and C_0 , together with the 'pure constant', were chosen in (226), and likewise for the coefficients of Σ_0 and S_0 in (296) already referred to. They were originally set as algebraic unknowns, and carried through the complete analysis for r_2 . The five unknowns were then derived, by solution of linear algebraic equations, so that the coefficients for Γ_1, Γ_{-1}, C_1, C_{-1} and $\cos v$ in (320) should all be zero.

6.3 Pure short-period perturbations in v and u

An approach similar to that used for r_2 would also be possible for v_2, but in this case the direct approach, only slightly modified, is simpler. The starting point is (42), which gives the partial derivatives v_e and v_M that occur in the first-order formula

$$v_1 = v_e e_1 + v_M M_1 \ . \qquad (321)$$

The corresponding formula for v_2 involves five terms, derived from the Taylor expansion as remarked in section 6.2, but differentiation of v_e and v_M with respect to M is not immediate, whereas with respect to v it is, so it is preferable to start with the following six-term formula:

$$v_2 = v_e e_2 + v_M M_2 + \tfrac{1}{2}(v_{ee} e_1^2 + v_{ev} e_1 v_1 + v_{Me} M_1 e_1 + v_{Mv} M_1 v_1) \ ; \qquad (322)$$

here

$$v_{ee} = q^{-4} \sin v \ [q^2 \cos v + 2e(2 + e \cos v)] \ , \qquad (323)$$

$$v_{ev} = - q^{-2} [e \sin^2 v - \cos v \ (2 + e \cos v)] \ , \qquad (324)$$

$$v_{Me} = q^{-5} (1 + e \cos v) [2q^2 \cos v + 3e(1 + e \cos v)] \qquad (325)$$

and

$$v_{Mv} = - 2eq^{-3} \sin v \ (1 + e \cos v) \ . \qquad (326)$$

(The hybrid nature of these partial derivatives must be noted; thus v_e is the e-derivative of v with M held constant, but v_{ee} is the e-derivative of v_e with v held constant.)

It is convenient, in substituting these second-order partial-derivative expressions into (322), to split their components into three combinations of terms. The first combination is given by the first half of each of v_{ee}, v_{ev} and v_{Me}, together with the whole of v_{Mv}; the terms involved are precisely the ones that arise when the differentiation of v_e and v_M, given by (42), is restricted to the variation of $e \cos v$, and this first combination of terms factorizes in a convenient way. The second combination involves the second half of v_{ee} and v_{ev}, and the third combination involves the second half of v_{Me}.

We now express the working formula for v_2, corresponding to (314) for r_2, as a sum of five terms, of which the first, third and fourth represent the three combinations of partial-derivative terms just described; the second term of the 'working formula' covers the e_2 term from (322), whilst its final term covers the M_2 term. Thus we get

$$\begin{aligned} v_2 = {} & \tfrac{1}{2}q^{-2}(e_1 \cos v - ev_1 \sin v)\left\{e_1 \sin v + 2q^{-1}M_1(1 + e \cos v)\right\} \\ & + q^{-2}e_2 \sin v\ (2 + e \cos v) \\ & + \tfrac{1}{2}q^{-4}e_1(2ee_1 \sin v + q^2v_1 \cos v)(2 + e \cos v) \\ & + \tfrac{3}{2}\, eq^{-5}e_1M_1(1 + e \cos v)^2 + q^{-3}M_2(1 + e \cos v)^2 . \end{aligned} \tag{327}$$

For the first term in (327), we have, from (148) and (182),

$$\begin{aligned} e_1 \cos v - ev_1 \sin v = {} & \tfrac{1}{24}\Big[f\left\{3e^2C_4 + 16eC_3 + 2[10,7]C_2 + 32eC_1 + 11e^2C_0\right\} \\ & + 4h\left\{e^2 \cos 2v + 8e \cos v + [6,1]\right\}\Big] , \end{aligned} \tag{328}$$

whilst from (148) and (164) we have

$$\begin{aligned} & e_1 \sin v + 2q^{-1}M_1(1 + e \cos v) = \\ & -\tfrac{1}{96}e^{-1}\Big[f\left\{3e^3S_6 + 30e^2S_5 + 10e[10,-1]S_4 + 2[56,-5]S_3 + 48eq^2S_2\right. \\ & \qquad \left. - 6[8,9]S_1 - 6e[14,1]S_0 - 30e^2S_{-1} - 3e^3S_{-2}\right\} \\ & \qquad + 4h\left\{e^3 \sin 4v + 10e^2 \sin 3v + 6e[6,-1] \sin 2v + 2[24,-7] \sin v\right\}\Big] . \end{aligned}$$

......(329)

Multiplying the product of these expressions by a further factor $\tfrac{1}{2}q^{-2}$, we get the required 'first term'; the result, in skeleton form, is

$$- \frac{1}{9216} e^{-1}q^{-2} \Big[f^2 \{ 9e^5\Sigma_{10} + \ldots - 33e^5\Sigma_{-2} \} + 8fh \{ 3e^5S_8 + \ldots - 7e^5S_{-4} \}$$

$$+ 2 \{ e^5(8h^2 + 21f^2) \sin 6v + \ldots$$

$$+ 4[16(72h^2 + 25f^2), 2(168h^2 + 457f^2), -(104h^2 + 33f^2)] \sin v \} \Big] .$$

For the second term of (327), we multiply out the expansion for e_2, *viz* (226), by $q^{-2} \sin v\ (2 + e \cos v)$. The result, in skeleton form is

$$- \frac{1}{36864} e^{-1}q^{-2} \Big[f^2 \{ 9e^5\Sigma_{12} + \ldots - 9e^5\Sigma_{-4} \} + 8f \{ 3e^5hS_{10} + \ldots - 3e^5hS_{-6} \}$$

$$+ 2 \{ e^5(8h^2 + 9f^2) \sin 8v + \ldots$$

$$- 16[4(216, -648, 56\text{[illegible]}), 24(44, -124, 77), (648, -312, -907)] \sin v \} \Big] .$$

Combination of the full versions of the first two terms of (327) now gives the following complete expression:

$$- \frac{1}{36864} e^{-1}q^{-2} \Big[f^2 \{ 9e^5\Sigma_{12} + 144e^4\Sigma_{11} + 6e^3[154, 19]\Sigma_{10} + 24e^2[124, 61]\Sigma_9$$

$$+ 2e[2408, 3592, 39]\Sigma_8 + 8[392, 2040, 133]\Sigma_7$$

$$+ 2e[7824, 2470, -859]\Sigma_6 + 32[98, 267, -314]\Sigma_5$$

$$+ 64e[46, -251, -68]\Sigma_4 - 32[18, 215, 416]\Sigma_3 - 6e[688, 882, 343]\Sigma_2$$

$$- 24[24, 192, -49]\Sigma_1 - 6e[312, 184, -115]\Sigma_0 - 24e^2[84, -11]\Sigma_{-1}$$

$$- 6e^3[138, -13]\Sigma_{-2} - 144e^4\Sigma_{-3} - 9e^5\Sigma_{-4} \}$$

$$+ 8f \{ 3e^5hS_{10} + 48e^4hS_9 + e^3[316h, -(98, -123)]S_8$$

$$+ 4e^2[268h, -(234, -291)]S_7$$

$$+ e[1872h, -4(670, -807), -(526, -717)]S_6$$

$$+ 4[336h, -2(130, -91), -(782, -1165)]S_5 +$$

$$+ e[16(266,-391),-2(914,-1787),-(346,-687)]S_4$$

$$+ 48[28h,3(62,-73),12(7,-6)]S_3$$

$$- 4e[8(78,-77),-(5074,-5785),-(422,-459)]S_2$$

$$- 16[36h,2(28,-23),-(518,-625)]S_1$$

$$+ 3e[16(26,-15),-2(2066,-2235),3(42,-71)]S_0$$

$$- 4[144h,2(766,-969),(2210,-2367)]S_{-1}$$

$$- e[1296h,1080(6,-7),(1418,-1503)]S_{-2}$$

$$- 4e^2[228h,(494,-561)]S_{-3} - e^3[300h,(190,-213)]S_{-4}$$

$$- 48e^4hS_{-5} - 3e^5hS_{-6}\}$$

$$+ 2\{e^4(8h^2 + 9f^2)(e \sin 8v + 16 \sin 7v)$$

$$+ 2e^3[6(72,-216,235),-(56,24,-171)] \sin 6v$$

$$+ 8e^2[24(16,-48,49),-(152,24,-465)] \sin 5v$$

$$+ 2e[24(120,-360,337),-8(280,-24,-925),-(392,-24,-661)] \sin 4v$$

$$+ 8[24(24,-72,61),-4(168,-24,-817),-(904,-504,-1211)] \sin 3v$$

$$+ 2e[16(72,-216,671),-2(4808,-5592,-4277),-(1368,-840,-1215)] \sin 2v$$

$$+ 32[6(24,-72,61),-(360,-984,89),-8(47,-39,-40)] \sin v\}] .$$

For the third term of (327) we require (to be multiplied out by $\frac{1}{2}q^{-4}e_1$) the following result, derived using (148) and (182) again:

$(2 + e \cos v)(2e\,e_1 \sin v + q^2 v_1 \cos v)$

$$= -\frac{1}{192} e^{-1} \Big[f\{3e^3[1,-3]S_7 + 30e^2[1,-3]S_6 + e[100,-283,-51]S_5 + 4[28,-53,-89]S_4 + 2e[122,-287,-45]S_3 + 64[1,3,-4]S_2 + 10e[6,35,1]S_1 - 4[12,-37,-89]S_0 - e[84,-251,-67]S_{-1} - 30e^2[1,-3]S_{-2} - 3e^3[1,-3]S_{-3}\} + 4h\{e^3[1,-3] \sin 5v + 10e^2[1,-3] \sin 4v + 3e[12,-35,-3] \sin 3v + 4[12,-31,-19] \sin 2v + 2e[18,-85,-3] \sin v\}\Big] . \qquad ......(330)$$

Again, the fourth term requires $3eq^{-1} M_1 (1 + e \cos v)^2$ to be multiplied by this same factor, $\frac{1}{2}q^{-4} e_1$. It is natural to combine the two terms, therefore, the result of the combination being expressible, with the factor $\frac{1}{2}q^{-4}$ included, as:

$$-\frac{1}{384} e^{-1}q^{-4}e_1 \Big[f\{3e^3S_7 + 30e^2S_6 + e[100, 53, -36]S_5 + 4[28,85,-56]S_4 + 2e[290,-119,-66]S_3 + 64[1,6,-7]S_2 - 2e[42,17,46]S_1 - 4[12,53,-8]S_0 - e[84,37,-4]S_{-1} - 30e^2S_{-2} - 3e^3S_{-3}\} + 4h\{e^3 \sin 5v + 10e^2 \sin 4v + 3e[12,5,-4] \sin 3v + 4[12,23,-16] \sin 2v + 2e[90,-49,-6] \sin v\} \Big] .$$

On substituting for e_1 from (148) and multiplying out, we get, in skeleton form,

$$-\frac{1}{36864} e^{-1}q^{-4} \Big[f^2\{9e^5\Sigma_{12} + \ldots - 9e^5\Sigma_{-4}\} + 8fh\{3e^5S_{10} + \ldots - 3e^5S_{-6}\} + 2\{e^5(8h^2 + 9f^2) \sin 8v + \ldots + 16[12(24h^2 + 7f^2), 2(984h^2 + 643f^2), -8(116h^2 - 43f^2), -(184h^2 + 427f^2)] \sin v\} \Big] .$$

This leaves just the last term of (327). On multiplying the expansion for M_2 , *viz* (296), by q^{-3} $(1 + e \cos v)^2$, we obtain for this last term, in skeleton form,

$$q^{-3} M_2 (1 + e \cos v)^2$$

$$= \frac{1}{36864} e^{-2} q^{-4} \Big[3f^2 \{ 3e^6 [2,-1] \Sigma_{12} + \ldots - 3e^6 [2,-1] \Sigma_{-4} \}$$

$$+ 8f \{ 3e^6 h [2,-1] S_{10} + \ldots - 3e^6 h [2,-1] S_{-6} \}$$

$$+ 2 \{ e^6 (8h^2 + 9f^2) [2,-1] \sin 8v + \ldots$$

$$+ 16e [48(16,-48,61), -24(0,64,-121),$$

$$6(64,16,-253), -(8,-312,453)] \sin v \} \Big] . \qquad (331)$$

On combining the full versions of (331) and the preceding expression (that represents the third and fourth terms of (327) together), a common factor q^2 can be extracted, for cancelling, and the resulting expression, in skeleton form, is

$$\frac{1}{36864} e^{-2} q^{-2} \Big[f^2 \{ 9e^6 \Sigma_{12} + \ldots - 9e^6 \Sigma_{-4} \} + 8f \{ 3e^6 h S_{10} + \ldots - 3e^6 h S_{-6} \}$$

$$+ 2 \{ e^6 (8h^2 + 9f^2) \sin 8v + \ldots$$

$$+ 32e [6(40,-120,183), -(744,-1464,307), -2(44,-60,97)] \sin v \} \Big] .$$

The grand combination of the full version of this expression, which represents the last three terms of (327), with the earlier expression (given in full) for the first two terms, is now possible. A further (common) factor of $8q^2$ emerges in the combination, and ten terms cancel out completely. The final result is

$$v_2 = \frac{1}{4608} e^{-2} \times$$

$$\times \Big[f^2 \{ 9e^4\Sigma_{10} + 108e^3\Sigma_9 + 6e^2[82,15]\Sigma_8 + 12e[84,67]\Sigma_7 + [784,2448,191]\Sigma_6$$

$$+ 16e[159,77]\Sigma_5 + 4e^2[482,25]\Sigma_4 - 48e[9,-8]\Sigma_3 - (144,624,-5)\Sigma_2$$

$$- 12e[36,19]\Sigma_1 - 6e^2[66,-1]\Sigma_0 - 108e^3\Sigma_{-1} - 9e^4\Sigma_{-2} \}$$

$$+ 4f \{ 6e^4hS_8 + 72e^3hS_7 + 2e^2[172h,-3(14,-17)]S_6$$

$$+ 12e[64h,-(34,-39)]S_5 + [672h,-36(2,3),-(94,-153)]S_4$$

$$+ 8e[8(21,-31),(130,-139)]S_3 + 2e^2[32(77,-92),(230,-243)]S_2$$

$$+ 32e[3(4,-3),(79,-91)]S_1 - [288h,16(116,-147),-(266,-303)]S_0$$

$$- 12e[48h,(110,-129)]S_{-1} - 6e^2[52h,(34,-39)]S_{-2}$$

$$- 72e^3hS_{-3} - 6e^4hS_{-4} \}$$

$$+ 2 \{ e^3(8h^2 + 9f^2)(e \sin 6v + 12 \sin 5v)$$

$$+ 6e^2[2(40,-120,127),-(8,8,-37)] \sin 4v$$

$$+ 12e[12(8,-24,23),-(24,24,-137)] \sin 3v$$

$$+ [48(24,-72,61),-96(4,8,-41),-(312,-168,-515)] \sin 2v$$

$$+ 8e[6(8,-24,61),-(144,-96,-257)] \sin v \} \Big] .$$

......(332)

With the formula for v_2 to hand, the formula for u_2 may be obtained at once, simply by incorporation of the formula for ϖ_2 , ie (273). The simplification is now dramatic, since 20 pairs of terms cancel in the combination, and a factor $2e^2$ can be extracted from the remaining terms. On the debit side, it is no longer the case that all the Σ-terms contain f^2 as a factor, whilst the S-terms contain f , but this property is restored when we finally proceed to w_2 in section 6.4. The result for u_2 is given by

$$u_2 = \frac{1}{2304}\Big[\{e^2(32,8,-43)\Sigma_6 + 4e(48,0,-53)\Sigma_5 + 8[2(18,-3,-17),(24,-15,-11)]\Sigma_4$$
$$+ 12e(48,-32,-19)\Sigma_3 + e^2(288,-264,-11)\Sigma_2\}$$
$$- 4\{2e^2(78,-158,81)S_4 + 4e(88,-198,119)S_3$$
$$- [8(48,-106,55),-(144,34,-153)]S_2 - 16e(102,-440,359)S_1$$
$$- 4e^2(18,-19,1)S_0\}$$
$$+ 4\{e^2(120,-60,-97)\sin 2v + 8e(156,-276,71)\sin v\}\Big] . \qquad (333)$$

The coefficients of Σ_j, S_j and $\sin jv$, for $|j| \le 2$, in both (332) and (333) are coloured by the choice of the 'constant' terms (for e_2 and M_2) in (226) and (296), that was made to suit r_2 , so it is noteworthy (and gratifying) that (333) contains no terms in Σ_1, Σ_{-1}, Σ_{-2}, S_{-1} or S_{-2} . No term in Σ_0 occurs, because it has been avoided by the choice of Σ_0 coefficient (for ω_2) in (273); there is still a term in S_0 , however, because the coefficient, in (273), was chosen to avoid a term in w_2 , not u_2 .

6.4 Pure short-period perturbations in b and w

We now have expressions for the pure second-order perturbations in r, u, i and Ω , given (in the form of r_2, u_2, i_2 and Ω_2 respectively) by (320), (333), (237) and (259). These are the four fundamental quantities of Kozai[17] and it remains, in pursuance of the philosophy of section 3.3, to compress u_2, i_2 and Ω_2 into formulae for b_2 and w_2 ; in combination with r_{2co}, b_{2co} and w_{2co} as given by (307)-(309), the spherical polar representation of second-order short-period perturbations will then be complete.

Our starting point is (90), which resulted from the identification of two formulae for $(x \quad y \quad z)^T$, one in terms of osculating elements and the other in terms of semi-mean elements. Making use of (87) and (91), we have, for the three components of (90),

$$\cos(\tilde{b} + \delta b)\cos(\tilde{w} + \delta w) = \cos\delta\Omega\cos u - c\sin\delta\Omega\sin u , \qquad (334)$$

$$\cos(\tilde{b} + \delta b)\sin(\tilde{w} + \delta w) = \bar{c}\sin\delta\Omega\cos u + \{\cos\delta i - c\bar{c}(1 - \cos\delta\Omega)\}\sin u$$

$$\ldots\ldots(335)$$

and

$$\sin(\tilde{b} + \delta b) = -\bar{s} \sin \delta\Omega \cos u + \{\sin \delta i + c\bar{s}(1 - \cos \delta\Omega)\} \sin u . \quad (336)$$

As these formulae include the long-period carry-over terms of the short-period perturbations, we have (ignoring J_3, of course, in the definition of the $\delta\zeta$)

$$\delta i = \bar{K} i_1 + \bar{K}^2 (i_2 + i_{2,\ell p} \bar{m}) \quad (337)$$

(and similarly for $\delta\Omega$); also

$$\delta b = \bar{K} b_1 + \bar{K}^2 (b_2 + b_{2co}) \quad (338)$$

(and similarly for δw). We have dealt with carry-over terms separately, however, so we can interpret (334)-(336) as applying to pure short-period perturbations, ignoring $i_{2,\ell p}$ in (337) and b_{2co} in (338). We also replace u in the three formulae by $\tilde{u} + \bar{K}u_1 + \bar{K}^2 u_2$.

It is now straightforward, in principle, to expand each side of the three formulae to $O(\bar{K}^2)$ and then to identify the expressions that emerge as coefficients of $\bar{K}^2$. (Identification of the coefficients of $\bar{K}$ merely leads again to the formulae (96) and (94) that were used in the generation of b_1 and w_1 , given by (189) and (190).) We naturally make use of the generic second-order formula

$$\sin\left(\theta_0 + K\theta_1 + K^2\theta_2\right) = \sin \theta_0 \left(1 - \tfrac{1}{2}K^2\theta_1^2\right) + \cos \theta_0 \left(K\theta_1 + K^2\theta_2\right) \quad (339)$$

and the corresponding cosine formula.

On this basis we derive from (336), since it is now permissible to omit bars from all the terms,

$$b_2 = \sin u \, i_2 - s \cos u \, \Omega_2 + \tfrac{1}{2} s c \sin u \, \Omega_1^2 + u_1 (\cos u \, i_1 + s \sin u \, \Omega_1) . \quad (340)$$

The first two terms of this formula are given by (237) and (259), and their combination is given by

$$\sin u\, i_2 - s \cos u\, \Omega_2$$

$$= \frac{1}{576} s\, c \Big\{ 2(1-f)\big(e^2 \sin(5u+2v) + 6e \sin(5u+v) + 3[3,2] \sin 5u + 18e \sin(4u+\omega) + 9e^2 \sin(3u+2\omega)\big)$$
$$+ \big(e^2(6,-13)\sigma_5 + 4e(15,-29)\sigma_4 + 6[(21,-41),3(2,-7)]\sigma_3 + 12e(15,-53)\sigma_2 + 54e^2(1,-1)\sigma_1\big)$$
$$- \big(3e^2(32,-41)s_3 + 8e(11,-18)s_2 - 4[3(9,-11),2(27,-35)]s_1 - 120e(13,-22)s_0 - 4e^2(21,-37)s_{-1}\big)\Big\}; \qquad \ldots\ldots(341)$$

the coefficients of σ_1, s_1 and s_{-1} here are determined by the choice of 'constants' made in (237) and (259), these choices being now close to explanation.

The first five terms of (341) are of the form $\sin(jv + 5\omega)$, but there is no point in having a special notation for these as they all disappear when the third term of (340) is combined with (341). This combination reduces to

$$\frac{1}{576} s\, c \Big\{ e^2(32,-39)\sigma_5 + 8e(18,-25)\sigma_4 + 6[4(6,-11),(16,-31)]\sigma_3$$
$$+ 24e(6,-25)\sigma_2 - 9e^2(20,-23)s_3 - 56e(2,-3)s_2$$
$$+ 8[3(6,-7),4(11,-13)]s_1 + 120e(14,-23)s_0 + 32e^2(3,-5)s_{-1}\Big\}.$$

It remains to evaluate the last term of (340). From (149) and (151) we get

$$\cos u\, i_1 + s \sin u\, \Omega_1 = \tfrac{1}{2} s\, c\, (2ec_2 + 3c_1) , \qquad (342)$$

and then multiplication by (183) yields the expression required. On combining this expression with the preceding expression for the first three terms of (340), we finally derive our result for b_2 . As with r_2 , given by (320), we express

this without use of the bracket conventions, its brevity being almost as remarkable as for r_2 ; thus,

$$b_2 = -\frac{1}{576} s c \left[f\{7e^2\sigma_5 + 40e\sigma_4 + 2(48 + 13e^2)\sigma_3 + 360e\sigma_2\} - \{e^2(140 - 177f)s_3 + 40e(8 - 9f)s_2 + 8e(120 - 229f)s_0\}\right] . \qquad \text{......(343)}$$

(In equation (43) of Ref 4, the factor $s\ c$ was unfortunately not included.)

This result is free of terms in σ_1, s_1 and s_{-1} , which would arise if the right choice had not been made for the coefficient of C_0 and the 'pure constant' in (237), and the coefficient of S_0 in (259). In practice, of course, the 'right' choice was made by setting algebraic unknowns in (237) and (259); these were carried through the analysis for b_2 and then determined such that the coefficients of σ_1, s_1 and s_{-1} would be zero.

To derive w_2 , we must return to (334) and (335), prepared to use either or both of these equations after elimination of $\cos(\bar{b} + \delta b)$ or substitution for its expansion as $(1 - \frac{1}{2}\bar{K}^2 b_1^2)$. Again recalling that the carry-over terms have already been dealt with, we can express these equations, after the aforesaid substitution and an expansion of the right-hand side, as

$$\cos w = \cos u - \bar{K}(\bar{c} \sin u\ \Omega_1) + \bar{K}^2\{\sin u\ (\bar{s} i_1 \Omega_1 - \bar{c}\Omega_2) - \tfrac{1}{2} \cos u\ (\Omega_1^2 - b_1^2)\} \qquad (344)$$

and

$$\sin w = \sin u + \bar{K}(\bar{c} \cos u\ \Omega_1) + \bar{K}^2\{\bar{c} \cos u\ \Omega_2 - \tfrac{1}{2} \sin u\ (i_1^2 + \bar{c}^2\Omega_1^2 - b_1^2)\} . \qquad (345)$$

It is now convenient to write, as in Refs 2 and 3,

$$\upsilon = w - u = O(K) , \qquad (346)$$

since (339) and the corresponding cosine formula then immediately give

$$\cos w = \cos u - \bar{K}\upsilon_1 \sin u - \bar{K}^2(\upsilon_2 \sin u + \tfrac{1}{2}\upsilon_1^2 \cos u) \tag{347}$$

and

$$\sin w = \sin u + \bar{K}\upsilon_1 \cos u + \bar{K}^2(\upsilon_2 \cos u - \tfrac{1}{2}\upsilon_1^2 \sin u)\ , \tag{348}$$

equations that may be identified with (344) and (345) respectively. In both identifications the terms in $\bar{K}$ simply lead to (94) in the form

$$\upsilon_1 = \bar{c}\Omega_1\ , \tag{349}$$

but the coefficients of $\bar{K}^2$ lead to two alternative expressions for υ_2, given by

$$(\upsilon_2 - \bar{c}\Omega_2) \sin u = -i_1\Omega_1\bar{s} \sin u - \tfrac{1}{2}\left(b_1^2 - \Omega_1^2 + \upsilon_1^2\right) \cos u \tag{350}$$

and

$$(\upsilon_2 - \bar{c}\Omega_2) \cos u = \tfrac{1}{2}\left(b_1^2 - \bar{c}^2\Omega_1^2 - i_1^2 + \upsilon_1^2\right) \sin u\ . \tag{351}$$

In Refs 2 and 3 it was thought best to develop (350) and (351) separately, showing how the same formula for $\upsilon_2 - \bar{c}\Omega_2$ could be derived from either, but the most natural procedure is to multiply the equations by $\sin u$ and $\cos u$ respectively, and then add them together. When this is done, we get, with the terms in b_1^2 (and also υ_1^2) cancelling as was inevitable,

$$\upsilon_2 - c\Omega_2 = -\tfrac{1}{4}\left\{2s(1 - \cos 2u)i_1\Omega_1 + \sin 2u\ (i_1^2 - f\Omega_1^2)\right\}, \tag{352}$$

since it is now legitimate to drop all bars.

The terms on the right-hand side of (352) can be obtained at once from (149) and (151). Thus, direct expansion of products gives

$$i_1\Omega_1 = \frac{1}{72} s(1-f)\{e^2\Sigma_6 + 6e\Sigma_5 + 3[3,2]\Sigma_4 + 18e\Sigma_3 + 9e^2\Sigma_2$$

$$- 6e^2S_4 - 12eS_3 + 6[3,-2]S_2 + 36eS_1 + 18e^2S_0 - 36e \sin v\} , \qquad (353)$$

$$i_1^2 = \frac{1}{72} f(1-f)\{e^2\Gamma_6 + 6e\Gamma_5 + 3[3,2]\Gamma_4 + 18e\Gamma_3 + 9e^2\Gamma_2 + 12eC_3 + 36C_2$$

$$+ 36eC_1 + 6e^2 \cos 2v + 24e \cos v + [27,10]\} \qquad (354)$$

and

$$\Omega_1^2 = -\frac{1}{72} (1-f)\{e^2\Gamma_6 + 6e\Gamma_5 + 3[3,2]\Gamma_4 + 18e\Gamma_3 + 9e^2\Gamma_2$$

$$- 12e^2C_4 - 36eC_3 - 24e^2C_2 + 36eC_1 + 36e^2C_0$$

$$+ 30e^2 \cos 2v - 24e \cos v - [9,46]\} , \qquad (355)$$

and the last two of these give

$$i_1^2 - f\Omega_1^2 = \frac{1}{36} f(1-f)\{e^2\Gamma_6 + 6e\Gamma_5 + 3[3,2]\Gamma_4 + 18e\Gamma_3 + 9e^2\Gamma_2$$

$$- 6e^2C_4 - 12eC_3 + 6[3,-2]C_2 + 36eC_1 + 18e^2C_0$$

$$+ 18e^2 \cos 2v + 9[1,-2]\} . \qquad (356)$$

It is now natural to combine the $\cos 2u$ component of the first term of (352) with the whole of the second term, the result being (with the factor $-\frac{1}{4}$ included)

$$-\frac{1}{144} f(1-f)(8e^2S_4 + 12eS_3 - 24e^2S_2 - 36eS_1 + 24e^2 \sin 2v + 48e \sin v) ,$$

after which the full evaluation of (352) yields

$$\upsilon_2 - \Omega_2 \cos i = -\frac{1}{144} f(1-f)\{e^2\Sigma_6 + 6e\Sigma_5 + 3[3,2]\Sigma_4 + 18e\Sigma_3 + 9e^2\Sigma_2 + 2e^2S_4$$

$$+ 18[1,-2]S_2 + 18e^2S_0 + 24e^2 \sin 2v + 12e \sin v\} . \qquad (357)$$

It now follows from (357) and (259) that

$$
\begin{aligned}
\upsilon_2 = -\frac{1}{288}(1-f)\Big\{ & e^2(4,5)\Sigma_6 + 24e(1,1)\Sigma_5 + 3[2(6,5),(8,3)]\Sigma_4 + 24e(3,1)\Sigma_3 \\
& + 3e^2(12,1)\Sigma_2 - 2e^2(39,-41)S_4 - 16e(11,-8)S_3 \\
& + 12[(16,-27),-2(3,4)]S_2 + 48e(17,-30)S_1 + 2e^2(18,-1)S_0 \\
& + 6e^2(4,13)\sin 2v + 48e(3,2)\sin v\Big\} .
\end{aligned}
$$

.......(358)

Finally, from (358), (346) and (333) it follows that

$$
\begin{aligned}
w_2 = -\frac{1}{2304}\Big[& f^2\{3e^2\Sigma_6 + 20e\Sigma_5 + 16(2+e^2)\Sigma_4 + 36e\Sigma_3 - 13e^2\Sigma_2\} \\
& + 4f\{2e^2(2-f)S_4 - 4e(46-55f)S_3 \\
& - [8(23-26f) + e^2(14-39f)]S_2 + 16e(158-179f)S_1\} \\
& - 4\{e^2(72-168f+59f^2)\sin 2v + 8e(120-264f+95f^2)\sin v\}\Big] .
\end{aligned}
$$

......(359)

As with (320) and (343), this has been deliberately written without the use of the bracket conventions. (In Ref 4 the wrong sign was attached to all the 4f terms.)

7 FIRST-ORDER ANALYSIS FOR J_3 PERTURBATIONS

7.1 Exact equations for rates of change of elements

The first-order analysis for J_3 is very similar to that for J_2 (section 4), the main differences being that (as for all the odd harmonics) there are no secular terms in the solution and that (as for all J_ℓ with $\ell > 2$) there are long-period perturbations induced by J_2. Our starting point is the set of exact equations for the variation of the osculating elements, given by substituting (49) into the planetary equations, (22)-(27). Working with (33) and (34), rather than (26) and (27), to keep the analysis (inevitably more laborious than for J_2) as simple as possible, we get

$$\dot{a} = \frac{Hnas}{12q^5}\left(\frac{p}{r}\right)^5\left\{5f(7e\gamma_4 + 6\gamma_3 - e\gamma_2) + 12g(5ec_2 + 2c_1 - 3ec_0)\right\}, \tag{360}$$

$$\dot{e} = \frac{Hns}{48q^3}\left(\frac{p}{r}\right)^4\left\{5f(7e\gamma_5 + 20\gamma_4 + 18e\gamma_3 + 4\gamma_2 - e\gamma_1)\right.$$
$$\left. + 12g(5ec_3 + 12c_2 + 6ec_1 - 4c_0 - 3ec_{-1})\right\}, \tag{361}$$

$$\dot{i} = \frac{Hnc}{4q^3}\left(\frac{p}{r}\right)^4(5f\gamma_3 + 4gc_1), \tag{362}$$

$$\dot{\Omega} = \frac{Hnc}{4q^3 s}\left(\frac{p}{r}\right)^4\left\{5f\sigma_3 + (4 - 15f)s_1\right\}, \tag{363}$$

$$\dot{\psi} = \frac{Hns}{48eq^3}\left(\frac{p}{r}\right)^4\left\{5f(7e\sigma_5 + 20\sigma_4 + 8e\sigma_3 - 4\sigma_2 + e\sigma_1)\right.$$
$$\left. + 12g(5es_3 + 12s_2 + 8es_1 + 4s_0 + 3es_{-1})\right\} \tag{364}$$

and

$$\dot{\rho} = \frac{2Hns}{3q^2}\left(\frac{p}{r}\right)^4(5f\sigma_3 + 12gs_1). \tag{365}$$

Integration of these equations is quite straightforward, on changing the independent variable from t to v (or rather $\bar{v}$) as in section 4. From the integration of each $\dot{\zeta}$ we obtain, in the notation of (72) and (73), the required expressions for $\zeta_{3,\ell p}$ and ζ_3. Results for ω, σ, $\int$, and hence M, are available at once from those for a, Ω, ψ and ρ, and the complete set of ζ_3 and $\zeta_{3,\ell p}$ are developed in sections 7.2 to 7.5. Formulae for r_3, b_3 and w_3 are obtained in section 7.7, based on the expressions derived for v_3 and u_3 in section 7.6. In Part 3 we make use of the $\zeta_{3,\ell p}$ in our study of the long-term variation of the mean orbital elements, as perturbed by J_2 and J_3, with singularity very much to the fore.

7.2 Perturbation in a

As in sections 4.1 and 5.1, the best way to evaluate a_3 is by the special method that appeals to the exact energy constant a'. On this basis, (52) leads, for the potential given by (49) alone, to

$$a/a' = 1 + \frac{1}{6} Hs(1 + e \cos v)^4 q^{-2} (5f\sigma_3 + 12gs_1) . \tag{366}$$

Thus

$$a_3 = \frac{1}{6} a q^{-2} s (p/r)^4 (5f\sigma_3 + 12gs_1) . \tag{367}$$

By combining successive powers of p/r ($= 1 + e \cos v$) with σ_3 and s_1 , we can, as in section 4.1, write down progressively more complicated alternative expressions for a_3 . These expressions, all of them necessarily symmetric in both their σ components and their s components, are

$$a_3 = \frac{1}{12} a q^{-2} s (p/r)^3 \{ 5f(e\sigma_4 + 2\sigma_3 + e\sigma_2) + 12g(es_2 + 2s_1 + es_0) \} , \tag{368}$$

$$\begin{aligned} a_3 = \frac{1}{24} a q^{-2} s (p/r)^2 \{ & 5f(e^2\sigma_5 + 4e\sigma_4 + 2[2,1]\sigma_3 + 4e\sigma_2 + e^2\sigma_1) \\ & + 12g(e^2s_3 + 4es_2 + 2[2,1]s_1 + 4es_0 + e^2s_{-1}) \} , \end{aligned} \tag{369}$$

$$\begin{aligned} a_3 = \frac{1}{48} a q^{-2} s(p/r) \{ & 5f(e^3\sigma_6 + 6e^2\sigma_5 + 3e[4,1]\sigma_4 + 4[2,3]\sigma_3 + 3e[4,1]\sigma_2 \\ & + 6e^2\sigma_1 + e^3\sigma_0) \\ & + 12g(e^3s_4 + 6e^2s_3 + 3e[4,1]s_2 + 4[2,3]s_1 + 3e[4,1]s_0 \\ & + 6e^2s_{-1} + e^3s_{-2}) \} \end{aligned}$$

......(370)

and

$$\begin{aligned} a_3 = \frac{1}{96} a q^{-2} s \{ & 5f(e^4\sigma_7 + 8e^3\sigma_6 + 4e^2[6,1]\sigma_5 + 8e[4,3]\sigma_4 + 2[8,24,3]\sigma_3 \\ & + 8e[4,3]\sigma_2 + 4e^2[6,1]\sigma_1 + 8e^3\sigma_0 + e^4\sigma_{-1}) \\ & + 12g(e^4s_5 + 8e^3s_4 + 4e^2[6,1]s_3 + 8e[4,3]s_2 + 2[8,24,3]s_1 \\ & + 8e[4,3]s_0 + 4e^2[6,1]s_{-1} + 8e^3s_{-2} + e^4s_{-3}) \} . \end{aligned}$$

......(371)

We shall use (369) for the development of M_3 via $\int_3$, (370) for the development of r_3 , and (371) for the development of p_3 .

The final expression, (371), can also of course be developed from (360) by the general method now to be used for e_3, i_3, etc.

7.3 Perturbations in e, i, p and pc^2

From (361), with the independent variable changed to v by use of (38), we get

$$\frac{de}{dv} = \frac{1}{192} Hs \left\{ 5f \left(7e^3\gamma_7 + 48e^2\gamma_6 + 4e[27,8]\gamma_5 + 16[5,9]\gamma_4 + 42e[4,1]\gamma_3 + 16[1,6]\gamma_2 + 4e[3,4]\gamma_1 - e^3\gamma_{-1} \right) + 12g \left(5e^3c_5 + 32e^2c_4 + 4e[17,4]c_3 + 16[3,4]c_2 + 14e[4,1]c_1 - 16q^2c_0 - 28ec_{-1} - 16e^2c_{-2} - 3e^3c_{-3} \right) \right\} . \quad \ldots\ldots(372)$$

The term in c_0 leads to the long-period variation specified by

$$e_{3,\ell p} = -q^2 g s \cos\omega , \quad (373)$$

whilst the rest of (372) yields, on integration,

$$e_3 = \frac{1}{192} s \left\{ f \left(5e^3\sigma_7 + 40e^2\sigma_6 + 4e[27,8]\sigma_5 + 20[5,9]\sigma_4 + 70e[4,1]\sigma_3 + 40[1,6]\sigma_2 + 20e[3,4]\sigma_1 + 40e^2\sigma_0 + 5e^3\sigma_{-1} \right) + 4g \left(3e^3s_5 + 24e^2s_4 + 4e[17,4]s_3 + 24[3,4]s_2 + 42e[4,1]s_1 + 24[1,6]s_0 + 84es_{-1} + 24e^2s_{-2} + 3e^3s_{-3} \right) \right\} ; \quad \ldots\ldots(374)$$

the coefficients of σ_0 and s_0 in (374) have been chosen to make r_3 , given by (408), as simple as possible.

From (362), in the same way, we get

$$\frac{di}{dv} = \frac{1}{16} Hc \left\{ 5f \left(e^2\gamma_5 + 4e\gamma_4 + 2[2,1]\gamma_3 + 4e\gamma_2 + e^2\gamma_1 \right) + 4g \left(e^2c_3 + 4ec_2 + 2[2,1]c_1 + 4ec_0 + e^2c_{-1} \right) \right\} . \qquad (375)$$

The term in c_0 leads to

$$i_{3,\ell p} = egc \cos \omega , \qquad (376)$$

whilst the rest of (375) yields

$$i_3 = \frac{1}{48} c \left\{ f \left(3e^2\sigma_5 + 15e\sigma_4 + 10[2,1]\sigma_3 + 30e\sigma_2 + 15e^2\sigma_1 \right) + 4g \left(e^2s_3 + 6es_2 + 6[2,1]s_1 - 3e^2s_{-1} \right) - 6e(4,-15)s_0 \right\} . \qquad (377)$$

The coefficient of s_0 in (377) has been chosen to make b_3, given by (410), as simple as possible; the fact that it is a multiple of $4 - 15f$, rather than $4 - 5f$ as for the other s coefficients, is bound up with the fact that in Ω_3, to be given by (385), the coefficient of c_0 is a multiple of $4 - 5f$ whilst the other c coefficients are multiples of $4 - 15f$. (See also Part 2.)

From (373), using (28), we get

$$p_{3,\ell p} = 2pegs \cos \omega , \qquad (378)$$

and from (371) and (374) we get

$$p_3 = \frac{1}{24} ps \left\{ f \left(3e^2\sigma_5 + 15e\sigma_4 + 10[2,1]\sigma_3 + 30e\sigma_2 + 15e^2\sigma_1 \right) + 4g \left(e^2s_3 + 6es_2 + 6[2,1]s_1 + 18es_0 - 3e^2s_{-1} \right) \right\} . \qquad (379)$$

We now have our usual check in terms of the quantity pc^2 that is an absolute constant of the motion. Equations (376) and (378) yield

$$(pc^2)_{3,\ell p} = 0 , \tag{380}$$

whilst (377) and (379) yield

$$(pc^2)_3 = \tfrac{1}{2} pesc^2 (8 - 15f) s_0 ; \tag{381}$$

(381) is 'constant' (to order zero), which completes the check, non-zero because the constants in e_3 and i_3 are not chosen by concomitant criteria.

Using (28) in reverse, we can get an expression for e_3 that is somewhat more compact than (374) but, like (173), is unfortunate in carrying the factor e^{-1} ; the expression, based on (367) and (379), is

$$e_3 = \frac{1}{48} e^{-1} s \Big\{ 4(p/r)^4 (5f\sigma_3 + 12gs_1) - q^2 f \big(3e^2\sigma_5 + 15e\sigma_4 + 10[2,1]\sigma_3 + 30e\sigma_2 + 15e^2\sigma_1\big) - 4q^2 g \big(e^2 s_3 + 6es_2 + 6[2,1]s_1 + 18es_0 - 3e^2 s_{-1}\big) \Big\} . \qquad \ldots\ldots(382)$$

7.4 Perturbations in Ω, ψ and ω

From (363) we get, after the change of integration variable,

$$\frac{d\Omega}{dv} = \frac{1}{16} Hcs^{-1} \Big\{ 5f\big(e^2\sigma_5 + 4e\sigma_4 + 2[2,1]\sigma_3 + 4e\sigma_2 + e^2\sigma_1\big) + (4 - 15f)\big(e^2 s_3 + 4es_2 + 2[2,1]s_1 + 4es_0 + e^2 s_{-1}\big) \Big\} . \tag{383}$$

The term in s_0 leads to

$$\Omega_{3,\ell p} = \tfrac{1}{4}e(4 - 15f)cs^{-1} \sin \omega , \tag{384}$$

whilst the rest of (383) yields

$$\Omega_3 = -\frac{1}{48} c s^{-1} \Big\{ f\big(3e^2\gamma_5 + 15e\gamma_4 + 10[2,1]\gamma_3 + 30e\gamma_2 + 15e^2\gamma_1\big) + (4 - 15f)\big(e^2c_3 + 6ec_2 + 6[2,1]c_1 - 3e^2c_{-1}\big) - 24eg\,c_0 \Big\} . \tag{385}$$

The coefficient of c_0 is chosen, in conjunction with the coefficient of s_0 in (377), to make b_3 as simple as possible; we have already remarked on the effective swapping of the coefficients $4 - 5f$ and $4 - 15f$, as between (377) and (385), and the full explanation for this (in the context of the general J_ℓ) is reserved for Part 2.

From (364) we get, after the change of integration variable,

$$\frac{d\Psi}{dv} = \frac{1}{192} He^{-1}s \Big\{ 5f\big(7e^3\sigma_7 + 48e^2\sigma_6 + 2e[54,11]\sigma_5 + 16[5,6]\sigma_4 + 24e[4,1]\sigma_3 - 16[1,-3]\sigma_2 - 2e[6,-5]\sigma_1 + e^3\sigma_{-1}\big) + 12g\big(5e^3s_5 + 32e^2s_4 + 2e[34,9]s_3 + 16[3,5]s_2 + 24e[4,1]s_1 + 16[1,4]s_0 + 14e[2,1]s_{-1} + 16e^2s_{-2} + 3e^3s_{-3}\big) \Big\} . \tag{386}$$

The term in s_0 leads to

$$\Psi_{3,\ell p} = e^{-1} g\, s\,(1 + 4e^2) \sin \omega , \tag{387}$$

whilst the rest of (386) yields

$$\Psi_3 = -\frac{1}{192}e^{-1}s\Big\{f\big(5e^3\gamma_7 + 40e^2\gamma_6 + 2e[54,11]\gamma_5 + 20[5,6]\gamma_4 + 40e[4,1]\gamma_3$$
$$- 40[1,-3]\gamma_2 - 10e[6,-5]\gamma_1 - 40e^2\gamma_0 - 5e^3\gamma_{-1}\big)$$
$$+ 4g\big(3e^3c_5 + 24e^2c_4 + 2e[34,9]c_3 + 24[3,5]c_2 + 72e[4,1]c_1$$
$$- 24c_0 - 42e[2,1]c_{-1} - 24e^2c_{-2} - 3e^3c_{-3}\big)\Big\};$$

......(388)

the coefficients of γ_0 and c_0 in (388) have been chosen to make w_3, given by (413), as simple as possible.

Expressions for $\omega_{3,\ell p}$ and ω_3 are now immediate, by subtraction of $c\Omega_{3,\ell p}$ from $\Psi_{3,\ell p}$ and of $c\Omega_3$ from Ψ_3. These expressions, which are necessarily more complicated than for $\Psi_{3,\ell p}$ and Ψ_3 because differing combinations of f are involved, are

$$\omega_{3,\ell p} = \frac{1}{4}e^{-1}s^{-1}\left[4fg - e^2(4 - 35f + 35f^2)\right]\sin\omega \qquad (389)$$

and

$$\omega_3 = -\frac{1}{192}e^{-1}s^{-1}\Big\{f\big(5e^3f\gamma_7 + 40e^2f\gamma_6 + 2e[54f,-(6,-17)]\gamma_5 + 20[5f,-3(1,-3)]\gamma_4$$
$$- 40e[2(1,-3),(1,-2)]\gamma_3 - 40[f,3(1,-2)]\gamma_2$$
$$- 10e[6f,(6,-11)]\gamma_1 - 40e^2f\gamma_0 - 5e^3f\gamma_{-1}\big)$$
$$+ 2\big(6e^3fgc_5 + 48e^2fgc_4 + e[136fg,-(8,-74,75)]c_3$$
$$+ 12[12fg,-(4,-39,40)]c_2 - 12e[2(4,-43,45),(4,-31,30)]c_1$$
$$- 48g[f,-(1,-1)]c_0 - 3e[56fg,-(8,-66,65)]c_{-1} - 48e^2fgc_{-2}$$
$$- 6e^3fgc_{-3}\big)\Big\}.$$

......(390)

7.5 Perturbations in ρ, σ, $\int$, M and L

From (365) we get, after the change of variable,

$$\frac{d\rho}{dv} = \frac{1}{6} Hqs \left\{ 5f\left(e^2\sigma_5 + 4e\sigma_4 + 2[2,1]\sigma_3 + 4e\sigma_2 + e^2\sigma_1\right) + 12g\left(e^2s_3 + 4es_2 + 2[2,\ 1]s_1 + 4es_0 + e^2s_{-1}\right) \right\}. \qquad (391)$$

The term in s_0 leads to

$$\rho_{3,\ell p} = 8eqsg \sin\omega , \qquad (392)$$

whilst the rest of (391) yields

$$\rho_3 = -\frac{1}{90}qs\left\{5f\left(3e^2\gamma_5 + 15e\gamma_4 + 10[2,1]\gamma_3 + 30e\gamma_2 + 15e^2\gamma_1\right) + 12g\left(5e^2c_3 + 30ec_2 + 30[2,1]c_1 + 12ec_0 - 15e^2c_{-1}\right)\right\}; \qquad (393)$$

the coefficients of γ_0 and c_0 (the former being zero) have been chosen to produce the appropriate coefficients (of γ_0 and c_0) in M_3 given by (400).

Expressions for $\sigma_{3,\ell p}$ and σ_3 are obtained, in view of (32), by subtraction of $q\psi_{3,\ell p}$ from $\rho_{3,\ell p}$ and $q\psi_3$ from ρ_3 , the results being

$$\sigma_{3,\ell p} = -e^{-1}qsg\ (1 - 4e^2)\sin\omega \qquad (394)$$

and

$$\begin{aligned}\sigma_3 = \frac{1}{2880}e^{-1}qs\ \Big\{&5f\left(15e^3\gamma_7 + 120e^2\gamma_6 + 6e[54,-5]\gamma_5 + 60[5,-2]\gamma_4\right.\\ &\left.- 40e[4,5]\gamma_3 - 120[1,5]\gamma_2 - 30e[6,11]\gamma_1 - 120e^2\gamma_0 - 15e^3\gamma_{-1}\right)\\ &+ 12g\left(15e^3c_5 + 120e^2c_4 + 10e[34,-7]c_3 + 360q^2c_2\right.\\ &- 120e[4,5]c_1 - 24[5,16]c_0 - 30e[14,-9]c_{-1} - 120e^2c_{-2}\\ &\left.- 15e^3c_{-3}\right)\Big\}.\end{aligned}$$

......(395)

For $\int_{3,\ell p}$ and $\int_3$ we proceed as in section 4.2, using the equation corresponding to (159), together with (369), to derive

$$\frac{n_3}{nW} = -\frac{1}{16} qs \left\{ 5f\left(e^2\sigma_5 + 4e\sigma_4 + 2[2,1]\sigma_3 + 4e\sigma_2 + e^2\sigma_1\right) + 12g\left(e^2s_3 + 4es_2 + 2[2,1]s_1 + 4es_0 + e^2s_{-1}\right) \right\}. \quad (396)$$

In view of (40), we now get

$$\int_{3,\ell p} = -3eqsg \sin\omega \quad (397)$$

and (on integrating the remaining terms)

$$\int_3 = \frac{1}{240} qs \left\{ 5f\left(3e^2\gamma_5 + 15e\gamma_4 + 10[2,1]\gamma_3 + 30e\gamma_2 + 15e^2\gamma_1\right) + 12g\left(5e^2c_3 + 30ec_2 + 30[2,1]c_1 + 12ec_0 - 15e^2c_{-1}\right) \right\}. \quad (398)$$

These results could have been obtained directly from (392) and (393), since $\int_{3,\ell p} = -\frac{3}{8}\rho_{3,\ell p}$ and $\int_3 = -\frac{3}{8}\rho_3$ from the general formula (quoted in section 4.2) with ℓ set to 3; the coefficient of c_0 in (398) has been taken to conform with this relation between ρ_3 and $\int_3$. (Taking different constants in $\int_3$ and $-\frac{3}{8}\rho_3$ would provide greater flexibility than we need; this was tacitly assumed also for $\int_2$ and $-\frac{1}{2}\rho_2$ in section 4.)

The formulae for σ and $\int$ can now be combined, the results of the combination being that

$$M_{3,\ell p} = -e^{-1}q^3sg \sin\omega \quad (399)$$

and

$$M_3 = \frac{1}{576} e^{-1} qs \left\{ f\left(15e^3\gamma_7 + 120e^2\gamma_6 + 6e[54,1]\gamma_5 + 60[5,1]\gamma_4 + 80eq^2\gamma_3 - 120[1,2]\gamma_2 - 30e[6,5]\gamma_1 - 120e^2\gamma_0 - 15e^3\gamma_{-1}\right) + 12g\left(3e^3c_5 + 24e^2c_4 + 2e[34,-1]c_3 + 72c_2 + 48eq^2c_1 - 24[1,2]c_0 - 6e[14,-3]c_{-1} - 24e^2c_{-2} - 3e^3c_{-3}\right) \right\}. \quad (400)$$

The perturbation in L need not detain us long. Thus, the formulae for $L_{3,\ell p}$ and L_3 follow at once from those for $\rho_{3,\ell p}$ and ρ_3 (or $\int_{3,\ell p}$ and $\int_3$) since (from the general formula of section 4.2, with ℓ set to 3)

$$L_{3,\ell p} = \frac{5}{8}\rho_{3,\ell p} = -\frac{5}{3}\int_{3,\ell p} = 5eqsg\sin\omega \tag{401}$$

and

$$L = \frac{5}{8}\rho_3 = -\frac{5}{3}\int_3 , \tag{402}$$

there being little point in quoting L_3 in full.

7.6 Short-period perturbations in v and u

It is recalled, from (74), that the (first-order) J_3 contribution to $\delta\zeta$ is given by $\bar{H}(\zeta_3 + \zeta_{3,\ell p}\bar{m})$. Also, δv and δu may be expressed in terms of $\delta\zeta$ by means of (42). Thus, from e_3, ω_3 and M_3 we may derive expressions for v_3 and u_3, whilst from $e_{3,\ell p}$, $\omega_{3,\ell p}$ and $M_{3,\ell p}$ we derive expressions for the quantities that, with an obvious notation referring to carry-over effects (cf section 6.1), we may denote by v_{3co} and u_{3co}. We have no need for a quantity $v_{3,\ell p}$, such that $v_{3co} = v_{3,\ell p}m$, because the long-period perturbations are applied (in a non-singular manner) to the orbital elements, and not to derived quantities.

From (42) it follows that δv can be developed as the sum of three components:

(A) $2q^{-2}(\delta e\sin v + eq^{-1}\,\delta M\cos v)$;

(B) $\frac{1}{2}eq^{-2}(\delta e\sin 2v + eq^{-1}\,\delta M\,[\cos 2v + 3])$; and

(C) $q^{-1}\,\delta M$.

We derive the three components separately for both the ζ_3 and the $\zeta_{3,\ell p}$, and then combine.

In combining the appropriate multiples of $e_3\sin v$ and $M_3\cos v$ to form Component A, eight pairs of) terms cancel, the result being

$$-\frac{1}{144}q^{-2}s\Big[f\big\{15e^3\gamma_6 + 60e^2\gamma_5 + e[28,47]\gamma_4 - 90q^2\gamma_3 - 5e[28,-13]\gamma_2$$
$$- 60e^2\gamma_1 - 15e^3\gamma_0\big\} +$$

$$+ 12g\{3e^3c_4 + 12e^2c_3 - e[4,-19]c_2 - 24q^2c_1 - 3e[4,1]c_0 - 12e^2c_{-1} - 3e^3c_{-2}\}] \; .$$

Rather more terms cancel, in a similar manner, in forming Component B, the result being

$$\frac{1}{288}\, eq^{-2}s\left[f\{30e^2\gamma_6 + 4e[37,-7]\gamma_5 + 15[13,-3]\gamma_4 + 96eq^2\gamma_3 - 15[1,9]\gamma_2 - 20\,e[1,5]\gamma_1 - 30e^2\gamma_0\}\right.$$
$$\left. + 24g\{3e^2c_4 + 6e[3,-1]c_3 + 3[8,-3]c_2 + 16eq^2c_1 - 15e^2c_0 - 6e[3,-1]c_{-1} - 3e^2c_{-2}\}\right] \; .$$

Four pairs of terms cancel in the combination of Components A and B, and a factor q^2 emerges to cancel the q^{-2}, the result being

$$\frac{1}{288}\, s\left[f\{20e^2\gamma_5 + 139e\gamma_4 + 12[15,8]\gamma_3 + 265e\gamma_2 + 100e^2\gamma_1\}\right.$$
$$\left. + 48g\{3e^2c_3 + 14ec_2 + 4[3,2]c_1 + 6ec_0 - 3e^2c_{-1}\}\right] \; .$$

Component C is given at once by (400), of course. It then follows, from the overall combination, that

$$v_3 = \frac{1}{576}\, e^{-1}s\left[f\{15e^3\gamma_7 + 120e^2\gamma_6 + 2e[162,31]\gamma_5 + 2[150,169]\gamma_4 + 8e[55,14]\gamma_3 - 10[12,-29]\gamma_2 - 10e[18,-5]\gamma_1 - 120e^2\gamma_0 - 15e^3\gamma_{-1}\}\right.$$
$$\left. + 12g\{3e^3c_5 + 24e^2c_4 + 2e[34,11]c_3 + 8[9,14]c_2 + 16e[9,1]c_1 - 24c_0 - 6e[14,1]c_{-1} - 24e^2c_{-2} - 3e^3c_{-3}\}\right] \; . \qquad \ldots\ldots(403)$$

From $e_{3,\ell p}$ and $M_{3,\ell p}$, it follows in the same way that Component A of the long-period carry-over is $-2sgms_1$, Component B is $-\frac{1}{2}esgm(s_2+3s_0)$, and Component C is $-e^{-1}q^2sgms_0$. Putting these together, we get

$$v_{3co} = -\tfrac{1}{2}e^{-1}sgm(e^2s_2 + 4es_1 + [2,1]s_0) . \qquad (404)$$

For δu, we simply feed in the effects of ω_3 and $\omega_{3,\ell p}$. Thus

$$\begin{aligned} u_3 = \frac{1}{288}s^{-1}\Big\{ & f\big(2e^2(9,-10)\gamma_5 + e(90,-101)\gamma_4 + 4[5(6,-7),(15,-16)]\gamma_3 \\ & \qquad + 5e(36,-43)\gamma_2 + 10e^2(9,-14)\gamma_1\big) \\ & + 6\big(e^2(4,-15,10)c_3 + 2e(12,-61,50)c_2 \\ & \quad + 2[6(4,-31,30),(12,-85,80)]c_1 - 6e(4,-9,5)c_0 \\ & \qquad - 3e^2(4,-31,30)c_{-1}\big)\Big\} \end{aligned} \qquad ......(405)$$

and

$$u_{3co} = -\frac{1}{8}s^{-1}m[4efgs_2 + 16fgs_1 + e(8,-66,65)s_0] . \qquad (406)$$

7.7 Short-period perturbations in r, b and w

We can now complete the J_3 analysis formally, by derivation of the quantities required in the spherical-coordinate representation, *viz* r_3, b_3 and w_3 (pure short-period terms), together with r_{3co}, b_{3co} and w_{3co} (long-period carry-over terms).

From (95) it follows that

$$r_3 = (r/a)\,a_3 - (a\cos v)\,e_3 + (aeq^{-1}\sin v)\,M_3 . \qquad (407)$$

Eight (pairs of) terms cancel in the combination of the terms in e_3 and M_3, the combination being

$$- \frac{1}{144} as \left\{ f \left(15e^3\sigma_6 + 90e^2\sigma_5 + e[176,49]\sigma_4 + 15[7,13]\sigma_3 + 5e[32,13]\sigma_2 \right.\right.$$

$$\left. + 90e^2\sigma_1 + 15e^3\sigma_0 \right)$$

$$+ 12g \left(3e^3s_4 + 18e^2s_3 + e[32,13]s_2 + 12[2,3]s_1 + 3e[16,-1]s_0 \right.$$

$$\left.\left. + 18e^2s_{-1} + 3e^3s_{-2} \right) \right\} .$$

On combining, further, with $(r/a)a_3$, given by (370), nine more terms disappear, with the terms in s_1, s_{-1}, σ_1 and σ_{-1} vanishing as a result of the choice of constants in (374) and (400), and a factor q^2 can be extracted from the rest. The result is

$$r_3 = \frac{1}{144} ps \left\{ f(4e\sigma_4 + 15\sigma_3 + 20e\sigma_2) + 48eg(s_2 - 3s_0) \right\} . \tag{408}$$

From $e_{3,\ell p}$ and $M_{3,\ell p}$, given by (373) and (395), we have, similarly,

$$r_{3co} = psgm \cos u . \tag{409}$$

Next we require b_3, and it follows from (96) that

$$b_3 = (\sin u)\, i_3 - (s \cos u)\, \Omega_3 . \tag{410}$$

Combination of the f-terms in (377) and (385) is immediately possible, giving

$$\frac{1}{48} cf \left(3e^2C_4 + 15eC_3 + 10[2,1]C_2 + 30eC_1 + 15e^2C_0 \right) .$$

The combination of the g-terms in (377) with the (4 - 15f) term in (385) is only a little more involved, and leads to two sets of terms, viz

$$- \frac{1}{12} c(2 - 5f) \left(e^2 \cos 2v - 3e \cos v - 3[2,1] \right)$$

and

$$-\frac{5}{48}\,c\,f\,\left(e^2C_4 + 6eC_3 + 6[2,1]C_2 - 3e^2C_0\right) .$$

Lastly, the chosen s_0 term in (377) combines with the chosen c_0 term in (385) to give

$$-\frac{1}{8}\,c\,e\left[5fC_1 + 2(2 - 5f)\cos v\right] ,$$

precisely the combination required to cancel the terms in C_1 and $\cos v$ already generated. Combination of the four sets of terms yields

$$b_3 = -\frac{1}{48}\,c\left\{f\left(2e^2C_4 + 15eC_3 + 20[2 + e^2]C_2 - 30e^2C_0\right) + 4(2 - 5f)\left(e^2\cos 2v - 3[2 + e^2]\right)\right\} . \qquad (411)$$

From $i_{3,\ell p}$ and $\Omega_{3,\ell p}$, given by (376) and (384), we have, similarly,

$$b_{3co} = \frac{1}{4}\,e\,c\,m\left[5fS_1 + 2(2 - 5f)\sin v\right] . \qquad (412)$$

Finally, we require w_3 and w_{3co}. Adding u_3, given by (405), to $c\Omega_3$, with Ω_3 given by (385), we have

$$w_3 = -\frac{1}{288}\,s\left\{f\left(2e^2\gamma_5 + 11e\gamma_4 + 4[5 + e^2]\gamma_3 + 35e\gamma_2 + 50e^2\gamma_1\right) - 24g\left(e^2c_3 - 2ec_2 - 2[18 + 7e^2]c_1 + 9e^2c_{-1}\right)\right\} . \qquad (413)$$

Equation (413) may also be obtained by direct addition of (403) and (388), representing v_3 and ψ_3 respectively. Yet another derivation is possible without using either u_3 or v_3; we return to the sum of the 'A' and 'B' components of v_3 derived in section 7.6; if to this sum we add $q^{-1}L_3$, as expressed by (402) via ρ_3 or $\int_3$, instead of $q^{-1}M_3$, then we have w_3 at once. From u_{3co} and $c\Omega_{3,\ell p}$, or from v_{3co} and $\psi_{3,\ell p}$, or most directly from the 'A' and 'B' components of v_{3co} together with $q^{-1}L_{3,\ell p}$, we obtain, similarly,

$$w_{3co} = -\tfrac{1}{2}\, sgm(es_2 + 4s_1 - 7es_0) \ . \qquad (414)$$

It will be observed, from (408), (411) and (413), that r_3, b_3 and w_3 contain five, six and nine terms respectively. Similarly, from (188)-(190), r_1, b_1 and w_1 contain two, two and three terms respectively. As an introduction to Part 2, we now give the number of terms associated with the general J_ℓ, noting that the quantities generically denoted by ζ_1 in the present Report will have to be relabelled ζ_2 in Part 2, where the analysis is taken to 'first order' only, but covers every ℓ. Odd and even values of ℓ have to be dealt with separately.

For odd ℓ, the number of terms is $\ell(\ell - \tfrac{3}{2}) + \tfrac{1}{2}$ for r_ℓ, $\ell(\ell - \tfrac{3}{2}) + \tfrac{3}{2}$ for b_ℓ, and ℓ^2 for w_ℓ. For even ℓ, the number is $\ell(\ell - \tfrac{3}{2}) + 1$ for both r_ℓ and b_ℓ, and $\ell^2 - 1$ for w_ℓ. That the last figure is $\ell^2 - 1$, rather than ℓ, is somewhat fortuitous, being due to the fact that (as we shall see in Part 2) the coefficient of $S_{\ell+1}$, ie of $\sin\{(\ell + 1)v + 2\omega\}$, which arises only when ℓ is even, always vanishes. Thus the absence of an S_3 term in (190), which seemed strange when the analysis was originally performed, was not a fluke occurring just for J_2.

8 CONCLUSIONS

The main function of this Report, Part 1 of a projected trilogy, has been to provide full details of the untruncated second-order orbital theory, involving J_2 and J_3 only, of which the outline was given in Ref 4. The principal novelty of this theory lies in the reduction of second-order J_2-induced perturbations to very compact expressions in a special system of spherical-polar coordinates based on a mean orbital plane. First-order J_3-induced perturbations, regarded as second-order in the overall theory, have been expressed in the same way, the results being particular cases of formulae (for the general J_ℓ) that will be developed in Part 2. (Postscript: now available as TR 89022.)

The special coordinate system cannot be used in the treatment of secular and long-period perturbations, but this is of no consequence as so few terms are present in the basic expressions for perturbations in the standard elliptic elements. The J_3 perturbations in these elements suffer from singularities, however, and the Report has included introductory material on the treatment of these singularities. Additional formulae relating to the long-term evolution of an orbit, subject to J_2 and J_3, will appear in Part 3 of the trilogy, which will also include numerical results from an assessment of the overall accuracy of

the second-order model for motion under J_2 and J_3. Over long periods of time, such that the total angle described by the satellite (in true or mean anomaly) approaches J_2^{-1} in order of magnitude, the errors in the model inevitably increase from third order to second order, *ie* typically (for close-Earth satellites) from centimetres to decametres.

The trilogy of Reports is to be viewed as part of a continuing study. The current limitations of the theory (for Earth satellites) may be summarized as follows: only perturbations due to the geopotential have been considered (though lunisolar perturbations were addressed in a precursor[5] to the present theory); tesseral harmonics have not been covered (though the effects of $J_{2,2}$ on a low-eccentricity orbit were considered in Ref 5); the zonal harmonics have been restricted to J_2 and J_3, except in Part 2 of the trilogy; the analysis has been taken to second order only, implying only first order in J_3; and the build-up of long-term error, as indicated in the last paragraph, cannot be avoided. In spite of these limitations, however, the present Report serves as a significant step towards the goal of an analytic (or semi-analytic) orbit generator that is much more efficient than numerical generators of comparable accuracy.

To facilitate a reference back to the formulae that summarize the formulae developed for the generator covering J_2 and J_3 only, we conclude the Report by recording the relevant equation numbers. The modified version of Kepler's third law is given by (299). Secular variation rates are given by (150) and (297) for Ω, and (152) and (298) for ω. Long-period variation rates are given by (300)-(306) for J_2^2 effects; for J_3 effects, the equations are (373), (376), (384), (387), (389), (392), (394), (397), (399) and (401). The carry-over effects on coordinates are given by (307)-(309) for J_2^2, and (409), (412) and (414) for J_3. Last, but far from least, the short-period effects on coordinates are given for J_2 (first order) by (188)-(190); for J_2^2, by (320), (343) and (359); and for J_3, by (408), (411) and (413).

LIST OF SYMBOLS

a	semi-major axis (osculating)
a'	energy-based fixed-mean semi-major axis
b	latitude-like coordinate of (r, b, w) system
c	$\cos i$ (also: obsolete cylindrical coordinate)
c_j	$\cos(jv + \omega)$
C_j	$\cos(jv + 2\omega)$
C_k	$\int_0^{\tau} \cos k\bar{\bar{\omega}}\, d\tau$
dn'	defined such that $n' + dn'$ constitutes $\dot{\int}_{sec}$
D	used in context of J_2^2 contributions to $d\zeta/d\bar{v}$
e	eccentricity
E	eccentric anomaly
f	$\sin^2 i$
F	Merson's functions, defined in equations (122)-(124)
g	$1 - \frac{5}{4} f$
h	$1 - \frac{3}{2} f$
H	$\frac{3}{2} J_3 (R/p)^3$
i	orbital inclination
j	arbitrary integer (usually associated with v)
J_ℓ	geopotential (zonal harmonic) coefficient of degree ℓ
k	arbitrary positive integer (usually associated with ω)
K	$\frac{3}{2} J_2 (R/p)^2$
ℓ	positive integer (suffix for J), usually 2 or 3
L	non-singular quantity such that $\dot{L} = \dot{M} + q\dot{\psi}$
m	$v - M$ (also: suffix, in $J_{\ell m}$, in section 1)
M	mean anomaly
n	mean motion
n'	energy-based fixed-mean mean motion
N	modified mean motion ($\bar{n}_L$ in Ref 4)

LIST OF SYMBOLS (continued)

p	semi-latus rectum ('parameter' of ellipse)
P_ℓ	Legendre-polynomial function of degree ℓ
q	$\sqrt{(1 - e^2)}$
Q_ζ	used as factor of $d\zeta/d\bar{v}$ in $J_2{}^2$ analysis
r	geocentric radius vector, coordinate of (r, b, w) system
r'	obsolete cylindrical coordinate
R	equatorial radius of the Earth (6378.14 km)
R_j	rotation matrix for j^{th} axis ($j = 1, 2, 3$)
R_j'	derived rotation matrix
s	$\sin i$ (*cf* c; and s_j, S_j, $\mathcal{S}_k$ by analogy)
t	time (measured from epoch)
T	affix for matrix transposition
u	argument of latitude $(v + \omega)$
u'	obsolete cylindrical coordinate
U	$M + \omega$
U'	quantity such that $\dot{U}' = \dot{M} + q\dot{\omega}$
$\mathcal{U}$	disturbing function ($\mathcal{U}_2$ and $\mathcal{U}_3$ as particular $\mathcal{U}_\ell$)
v	true anomaly
w	longitude-like coordinate of (r, b, w) system
$\dot{w}'$	$\dot{w} + \bar{c}\dot{\bar{\Omega}}_{\ell p}$
W	$q^{-3}(p/r)^2$
x, y, z	usual (equator based) geocentric coordinates
X, Y, Z	geocentric coordinates based on the mean orbital plane
β	geocentric latitude (*ie* declination)
γ_j	$\cos(j\tilde{v} + 3\tilde{\omega})$
Γ_j	$\cos(j\tilde{v} + 4\tilde{\omega})$
$\delta\zeta$	total short-period perturbation (in ζ)

LIST OF SYMBOLS (continued)

$\delta_p\zeta$	pure (Poisson) short-period perturbation
$\delta\zeta$	perturbation relative to $\tilde{\zeta}$
$\Delta\zeta$	long-period perturbation (from epoch)
$\nabla\zeta$	$\dot{\zeta}_{sec}\,t$ (used when ζ is ω)
ζ	generic orbital element (osculating)
$\bar{\zeta}$	mean ζ (notation used in $\bar{r}$ *etc* also)
$\bar{\bar{\zeta}}$	$\bar{\zeta} - \Delta\zeta$ (when ζ is ω, in particular)
$\tilde{\zeta}$	semi-mean ζ
$\dot{\zeta}_{\ell p}$	long-period component of $\dot{\zeta}$
$\dot{\zeta}_{sec}$	secular component of $\dot{\zeta}$
$\hat{\zeta}$	$\dot{\zeta}_{sec}/\bar{n}$
ζ_0	value of ζ at epoch (likewise $\bar{\zeta}_0$, more usually required)
ζ_j	cofactor of $\bar{K}$, $\bar{K}^2$, $\bar{H}$ ($j = 1, 2, 3$) in $\delta_p\zeta$
$\zeta_{j,\ell p}$	cofactor (likewise) in $\dot{\zeta}_{\ell p}/\bar{n}$
$\hat{\zeta}_j$	cofactor (likewise) in $\hat{\zeta}$
ζ	see ξ
η	$e \sin \omega$
η	see ξ
θ	arbitrary angular quantity
μ	Earth's gravitational constant (398 600.4 km^3/s^2)
$\hat{\mu}_1$, $\hat{\mu}_2$	cofactors of $\bar{K}$ and $\bar{K}^2$ in Kepler's third law (modified)
ξ	$e \cos \omega$
ξ, η, ζ	$\sin i \sin \Omega$, $-\sin i \cos \Omega$, $\cos i$
ρ	quantity such that $\dot{\rho} = \dot{\sigma} + q\dot{\psi}$
σ	quantity such that $M = \sigma + \int$
σ_j, Σ_j	as γ_j, Γ_j, but sines
τ	time (dummy variable for t in integrals)
υ	$w - u$

LIST OF SYMBOLS (concluded)

ϕ	quantity such that $e = \sin\phi$
ψ	quantity such that $\dot{\psi} = \dot{\omega} + c\dot{\Omega}$
ω	argument of perigee
$\bar{\bar{\omega}}$	see $\bar{\bar{\zeta}}$
$\tilde{\omega}$	$\omega \pm \Omega$
Ω	right ascension of the ascending node
$\int$	$\int_0^t n\,d\tau$
(, ,)	bracket convention for polynomial in f
[, ,]	bracket convention for polynomial in e^2

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